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A COMBINED NEWTON-RAPHSON AND GRADIENT PARAMETER CORRECTION TECHNIQUE FOR SOLUTION OF OPTIMAL-CONTROL PROBLEMS

by Ernest S. Armstrong

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SUMMARY

A parameter correction technique is developed to solve a boundary-value problem which frequently occurs in optimal-control theory. It is assumed that an indirect optimal-control method has been applied to a controllable dynamic system with a two-point boundary-value problem resulting such that the boundary conditions take the form of a set of unknown parameters to be determined to meet an equal number of terminal conditions. The optimal-control law is a piecewise continuous function with discontinuities occurring only at the zeros of certain continuous functions. A procedure is developed to improve upon an assumed set of parameters so that, by repetitive use of a correction formula, a monotonic decreasing sequence of values of a positive definite function that measures the terminal errors is produced. The direction of the correction vector is found to lie between the directions given by the gradient and the Newton-Raphson procedures.

Integral equations are derived for influence matrices that describe the effect of a change in the parameters on the terminal conditions.

The procedure is successfully applied to the determination of both planar and nonplanar fuel-optimal trajectories for a space vehicle which is launched from the surface of the moon and required to rendezvous with a space vehicle in a circular orbit.

INTRODUCTION

In recent years, control theory has been expanded to include the area of system optimization. This expansion has brought about a new design philosophy. Control

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*This report is based in part upon a thesis entitled "An Algorithm for the Iterative Solution of a Class of Two-Point Boundary-Value Problems Occurring in Optimal-Control Theory" offered in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics, North Carolina State University of Raleigh, Raleigh, North Carolina, June 1967.

functions may now be chosen to optimize in some given sense the system response to the control action; for example, a control law may be found to force a system to a given state while some functional of the system variables is minimized. This new area of research is termed optimal-control theory.

Present-day methods of calculating optimal-control solutions can be grouped into two classes: direct and indirect. Both methods are designed to minimize the value of some functional. A direct method depends upon a comparison of the values of the functional at two or more points. An indirect method is used to find a solution by means of necessary (and sometimes sufficient) conditions for a minimum. Typical direct methods are contained in references 1 to 3. Necessary conditions to be used in an indirect approach are found by applying the Pontryagin maximum principle (ref. 4), or the calculus of variations (ref. 5), or dynamic programing (ref. 6). In general, the necessary conditions take the form of a set of nonlinear differential equations with both initial and final boundary conditions; that is, in order to obtain explicit solutions, a nonlinear two-point boundary-value problem must be solved.

The advent of high-speed computers has made feasible the solution of optimization problems by the method of successive approximations. This procedure is markedly illustrated by the success of the aforementioned direct methods. In these methods, a control history is first assumed and then successively improved upon by the computation of time-dependent corrections arrived at through the use of gradient (refs. 1 and 2) or conjugate-gradient (ref. 3) theory in function space. Although many useful results have been obtained in this manner, direct methods, in general, do not guarantee that the solutions obtained satisfy the necessary conditions of the indirect theory.

A more rigorous, but computationally more difficult, approach is the use of necessary conditions of the indirect theory for the generation of optimal results. In this way, one of the theories of references 4, 5, or 6 is applied, and then an attempt is made to solve whatever boundary-value problem might ensue. This approach is adopted herein.

The purpose of this report is to present a successive approximation procedure for attacking a class of two-point boundary-value problems which frequently occurs in the application of indirect optimization theory. Basically, the boundary-value problem is one in which the optimal-control law is piecewise continuous and in which there are a number of system parameters to be determined to meet an equal number of terminal conditions. A parameter correction procedure is developed in which an assumed set of parameters can be improved upon so that, by repetitive use of a correction formula, a monotonic decreasing sequence of values of a positive definite function that measures the terminal errors is produced. The direction of the parameter correction vector lies between the direction given by the gradient and the Newton-Raphson procedures (ref. 7).

Integral equations are derived, the solutions of which yield influence matrices that describe the effect of a change in the parameters on the terminal conditions.

In order to exemplify the usefulness of the procedure, the Pontryagin maximum principle is applied to determine planar and nonplanar fuel-optimal trajectories for a space vehicle which is launched from the surface of the moon and required to rendezvous with a space station in a circular orbit. The technique is then successfully applied to solve the resulting two-point boundary-value problem.

SYMBOLS

A,D,M,N,K,L,S,G constant matrices

b_i positive weighing elements $(i = 1, 2, \ldots m)$

B m-dimensional diagonal matrix with elements b_i

$\sqrt{B}$ m-dimensional diagonal matrix with elements $\sqrt{b_i}$

c effective exhaust velocity

$\bar{c} = -A \frac{\partial \bar{e}'}{\partial \bar{\alpha}}\left(\bar{\alpha}^o, t_f\right) B \bar{e}\left(\bar{\alpha}^o, t_f\right)$

c_i elements of $\bar{c}$ $(i = 1, 2, \ldots m)$

C used as $a(t)\big|_C$ to designate continuous part of function $a(t)$

$d = \psi_2\left(x_1 + R_{s_x}\right) + \psi_4\left(x_3 + R_{s_y}\right) + \psi_6\left(x_5 + R_{s_z}\right)$

$\bar{e}$ m-dimensional vector with elements e_i

e_i terminal errors $(i = 1, 2, \ldots m)$

$E\left(\bar{\alpha}, t_f\right) = \frac{\bar{e} \cdot B\bar{e}}{2} = \sum_{i=1}^{m} \frac{b_i e_i^2}{2}$

$\bar{f}$ n-dimensional column vector with elements f_i

f_i function introduced in equation (1a) $(i = 1, 2, \ldots n)$

$f_o(\bar{x},\bar{u})$ integrand of F

F	function to be minimized, $\int_{t_o}^{t_f} f_o(\bar{x},\bar{u})dt$
$\bar{g}$	n-dimensional column vector with elements defined in equation (2)
g_i	elements of vector $\bar{g}$ $(i = 1, 2, \ldots n)$
$\bar{h} = \mathrm{col}(\psi_1,\psi_2, \ldots \psi_6)$	
$\underset{\sim}{H} = \sum_{k=0}^{n} \psi_k \dot{x}_k$	
$H(a) = \frac{1}{2}(1 + \mathrm{sgn}\ a)$	
$\hat{i},\hat{j},\hat{k}$	unit vectors
I	identity matrix
$J(t)$	set difference, $[t_o,t] - S(t^*)$
$\bar{k} = \begin{pmatrix} \delta\bar{\alpha}^o \\ \beta \end{pmatrix}$	
l	total number of switching functions
m	number of unknown parameters and terminal conditions
$m(t)$	total vehicle mass
m_o	initial mass of launch vehicle
max	maximum
$\underset{\sim}{M}$	maximum value of $\underset{\sim}{H}$ with respect to choice of $\bar{u}$
n	dimension of $\bar{x}$ and $\bar{\psi}$ in equation (2)
p	smallest positive integer where $\rho^{(p)}(\bar{\alpha},t) \neq 0$

r dimension of $\bar{u}$

$\bar{r} = \bar{R}_v - \bar{R}_s$

r_x, r_y, r_z elements of $\bar{r}$

$\dot{r}_x, \dot{r}_y, \dot{r}_z$ elements of $\dot{\bar{r}}$

R_s satellite orbital radius about moon, $\left(\bar{R}_s \cdot \bar{R}_s\right)^{1/2}$

$\bar{R}_s = \hat{i} R_{s_x} + \hat{j} R_{s_y} + \hat{k} R_{s_z}$

$R_{s_x}, R_{s_y}, R_{s_z}$ elements of $\bar{R}_s$

$\dot{R}_{s_x}, \dot{R}_{s_y}, \dot{R}_{s_z}$ elements of $\dot{\bar{R}}_s$

$\bar{R}_s^{\,o}, \dot{\bar{R}}_s^{\,o}$ initial values of $\bar{R}_s$ and $\dot{\bar{R}}_s$, respectively

R_v magnitude of position vector of interceptor vehicle, $\left(\bar{R}_v \cdot \bar{R}_v\right)^{1/2}$

$\bar{R}_v = \hat{i} R_{v_x} + \hat{j} R_{v_y} + \hat{k} R_{v_z}$

$R_{v_x}, R_{v_y}, R_{v_z}$ elements of $\bar{R}_v$

$\dot{R}_{v_x}, \dot{R}_{v_y}, \dot{R}_{v_z}$ elements of $\dot{\bar{R}}_v$

$\bar{R}_v^{\,o}, \dot{\bar{R}}_v^{\,o}$ initial values of $\bar{R}_v$ and $\dot{\bar{R}}_v$, respectively

s dummy integration variable

sgn a general signum function defined by $\begin{cases} 1 & (a > 0) \\ -1 & (a < 0) \\ \text{Unspecified} & (a = 0) \end{cases}$

sgn ρ_i a particular signum function defined in equation (4)

$S(t^*)$ set of switching points of all switching functions ρ_q $(q = 1, 2, \ldots l)$

$S_q(t^*)$ set of switching points of switching function ρ_q

t element of $\left[t_o, t_f\right]$

t_f	final time
t_o	initial time
t^+,t^-	t approached through values larger than t and smaller than t, respectively
t^*	arbitrary switching point
t_i^*	ith switching point in $S(t^*)$
T	magnitude of thrust-control vector
$\bar{T}$	thrust-control vector
T_1	constant matrix defined in equation (B4)
$T_2(t)$	matrix defined in equation (B5)
$\bar{u}$	r-dimensional vector with elements u_i
$\hat{u}$	unit vector using some elements of $\bar{u}$
u_i	control elements $(i = 1, 2, \ldots r)$
U	r-dimensional Euclidean space containing $\bar{u}$
v	total number of switching points in $S(t^*)$

$\bar{v} = \text{col}\left(x_1, x_2, \ldots x_6\right)$

$\bar{v}_o$	initial value of $\bar{v}$
V	n-dimensional Euclidean space containing $\bar{x}$
x,y,z	coordinates of axis system in figure B-1

$\sqrt{x} = \sqrt{\left(x_1 + R_{s_x}\right)^2 + \left(x_3 + R_{s_y}\right)^2 + \left(x_5 + R_{s_z}\right)^2}$

$\bar{x}$	vector with elements x_i

x_i — state variables given by equation (1a) $(i = 1, 2, \ldots n)$

$x_o = \int_{t_o}^{t_f} f_o(\bar{x},\bar{u})ds$

x',y',z' — coordinates of axis system in figure B-3

$\bar{x}^o = \bar{x}(t_o)$

$\bar{x}^1 = \bar{x}(t_f)$

X,Y,Z — coordinates of axis system in figure B-3

$X(\bar{\alpha},t_f)$ — equation (15) evaluated at $(\bar{\alpha},t_f)$

$\bar{Y}$ — vector defined in equation (B11)

$\bar{\alpha}$ — m-dimensional vector with elements α_i

α_i — unknown parameters $(i = 1, 2, \ldots m)$

$\bar{\alpha}^i$ — vector defined in theorem 1 $(i = 0, 1, \ldots)$

β — variable converting $\delta\bar{\alpha} \cdot \delta\bar{\alpha} \leqq \nu^2$ into $\delta\bar{\alpha} \cdot \delta\bar{\alpha} - \nu^2 + \beta^2 = 0$

γ — largest allowable value of thrust magnitude

$\delta a,\Delta a$ — increment in a

$\delta\bar{\alpha}_G$ — increment in $\bar{\alpha}$ in gradient direction of $-E(\bar{\alpha},t_f)$

$\delta\bar{\alpha}_{NR}$ — increment in $\bar{\alpha}$ in Newton-Raphson direction

$\delta\bar{v} = A\delta\bar{\alpha}^o$

$\Delta E(\bar{\alpha},t_f) = E(\bar{\alpha} + \delta\bar{\alpha},t_f) - E(\bar{\alpha},t_f)$

$\tilde{\Delta}E(\bar{\alpha},t_f)$ — defined by equation (7)

$\bar{\xi},\bar{\eta}$ — arbitrary m-dimensional vectors

θ_c, φ_c	angles defined in figure B-2
θ_v^o, φ_v^o	angles defined in figure B-4
$\iota_o, \theta_o, \varphi_o$	angles defined in figure B-3
λ	Lagrange multiplier
$\lambda(\nu)$	value of λ associated with ν
λ_i	ith eigenvalue of matrix $\frac{\partial \bar{e}'}{\partial \bar{\alpha}} B \frac{\partial \bar{e}}{\partial \bar{\alpha}}$ (i = 1, 2, . . . m)
μ	gravitational parameter of moon
ν	bound on magnitude of $\delta\bar{\alpha}$
ν_i	bound on magnitude of $\delta\bar{\alpha}^i$
$\bar{\rho}$	l-dimensional column vector of elements ρ_i
ρ_i	switching function (i = 1, 2, . . . l)
τ	arbitrary value of $t \epsilon \left[t_o, t_f\right]$
$\varphi\left(\bar{x}, \bar{\psi}, \bar{\alpha}, t\right)$	scalar function used as stopping condition

$\sqrt{\psi} = \sqrt{{\psi_2}^2 + {\psi_4}^2 + {\psi_6}^2}$

$\bar{\psi}$	vector with elements ψ_i (i = 1, 2, . . . n)
ψ_i	variables introduced by Pontryagin maximum principle (i = 0, 1, 2, . . . n)
$\Psi\left(\bar{\alpha}, t_f\right)$	equation (17) evaluated at $\left(\bar{\alpha}, t_f\right)$
ω	angular velocity of moon about axis of rotation
$\bar{\omega} = \hat{k}\omega$	
Ω	angular velocity of target in orbital plane
$\lvert a \rvert$	absolute value of a

$\bar{a}$ designates a is vector

$\|\bar{a}\| = (\bar{a} \cdot \bar{a})^{1/2}$

$[a,b]$ closed interval

(a,b) open interval

$\dot{a}(t)$ first derivative of $a(t)$ with respect to t

$\ddot{a}(t)$ second derivative of $a(t)$ with respect to t

$a^{(n)}(t), \dfrac{d^n a(t)}{dt^n}$ nth derivative of $a(t)$ with respect to t

$\bar{a} \cdot \bar{b} = \sum_{i=1}^{n} a_i b_i$ if $\bar{a} = \mathrm{col}(a_1, \ldots a_n)$ and $\bar{b} = \mathrm{col}(b_1, \ldots b_n)$

$\sum\limits_{\tilde{\rho}_q(t^*)}^{1-l}$ sum over all switching functions ρ_q $(q = 1, 2, \ldots l)$ which have switching points at t^*

$\dfrac{\partial \bar{x}}{\partial \bar{y}}$ $M \times N$ Jacobian matrix with elements $\dfrac{\partial x_i}{\partial y_j}$ where $i = 1, 2, \ldots M$ and $j = 1, 2, \ldots N$ if $\bar{x} = \mathrm{col}(x_1, \ldots x_M)$ and $\bar{y} = \mathrm{col}(y_1, \ldots y_N)$

$\dfrac{\partial^2 a}{\partial \bar{y}\ \partial \bar{x}}$ $M \times N$ matrix with elements $\dfrac{\partial^2 a}{\partial x_i\ \partial y_j}$ where $i = 1, 2, \ldots M$ and $j = 1, 2, \ldots N$ if $\bar{x} = \mathrm{col}(x_1, \ldots x_M)$ and $\bar{y} = \mathrm{col}(y_1, \ldots y_N)$

$\bar{0}$ null vector

ϵ belongs to a set

$\triangleq$ defined by

$\hat{}$ over a variable indicates a function of $\bar{\alpha}$ and t obtained by substitution for $\bar{x}(\bar{\alpha},t)$ and $\bar{\psi}(\bar{\alpha},t)$ into a function of $\bar{x}(\bar{\alpha},t)$, $\bar{\psi}(\bar{\alpha},t)$, $\bar{\alpha}$, and t

$\sim$ over a variable indicates a function of t obtained by substitution for $\bar{x}(\bar{\alpha},t)$ $\bar{\psi}(\bar{\alpha},t)$, and $\bar{\alpha}$ in a function of $\bar{x}(\bar{\alpha},t)$, $\bar{\psi}(\bar{\alpha},t)$, $\bar{\alpha}$, and t

$'$ over a matrix denotes matrix transpose

Subscripts:

j,k,m,n,q,r,v,M,N integers

Superscripts:

i,p integers

PROBLEM STATEMENT

Indirect Optimal-Control Theory

Consider the behavior of a dynamical system the state of which at any instant of time is characterized by n variables $x_1, x_2, \ldots x_n$. For example, these variables might represent the position and velocity coordinates of a space vehicle. The behavior of the system is simply the time variation of the vector variable $\bar{x} = \mathrm{col}\left(x_1, x_2, \ldots x_n\right)$, commonly referred to as the state vector. The vector space V of the vector variable $\bar{x}$ is termed the state space of the system.

Assume that the state of the system can be controlled; that is, a set of system inputs, the manipulation of which governs the state, are available. Assume that there are r such controls and that they are characterized by a point $\bar{u}$ in a region U of r-dimensional Euclidean space. For the purposes of this report, $\bar{u}(t) = \mathrm{col}\left[u_1(t), \ldots u_r(t)\right]$ is said to be admissible if each component is a piecewise continuous function such that $\bar{u}(t)$ belongs to U for each t $\left(t_o \leqq t \leqq t_f\right)$. The initial time t_o is assumed fixed, but the final time t_f may be either free or fixed. Let the behavior of the dynamical system be characterized by the autonomous differential equations

$$\frac{dx_i}{dt} = f_i\left(x_1, x_2, \ldots x_n, u_1, u_2, \ldots u_r\right) \qquad (i = 1, 2, \ldots n)$$

In vector form

$$\frac{d\bar{x}}{dt} = \bar{f}(\bar{x},\bar{u})$$

where $\bar{f}(\bar{x},\bar{u})$ is a vector function of $\bar{x}$ and $\bar{u}$. The functions f_i, for every $\bar{x} \epsilon V$ and $\bar{u} \epsilon U$ are assumed to be continuous with respect to all variables $x_1, \ldots x_n$ and $u_1, \ldots u_r$. Also the functions f_i are continuously differentiable with respect to $x_1, \ldots x_n$; that is,

$$f_i(\bar{x},\bar{u}) \text{ and } \frac{\partial f_i}{\partial x_j}(\bar{x},\bar{u}) \qquad (i = 1, 2, \ldots n;\ j = 1, 2, \ldots n)$$

are defined and continuous for all $\bar{x} \epsilon V$ and $\bar{u} \epsilon U$.

The admissible control $\bar{u}(t)$ is said to transfer the point $\bar{x}^o$ to the point $\bar{x}^1$ if the solution $\bar{x}(t)$ of $\frac{d\bar{x}}{dt} = \bar{f}(\bar{x},\bar{u})$ $\left(\bar{x}(t_o) = \bar{x}^o\right)$ passes through the point $\bar{x}^1$ at a certain instant of time t_f; that is, $\bar{x}(t_f) = \bar{x}^1$.

The quality of system performance is now assumed to be measured in terms of some functional

$$F = \int_{t_o}^{t_f} f_o\left[\bar{x}(t),\bar{u}(t)\right]dt$$

For example, the smaller the value of F, the better the system behaves. The scalar function $f_o\left[\bar{x}(t),\bar{u}(t)\right]$ is defined and differentiable with respect to x_i $(i = 1, 2, \ldots n)$.

The optimization problem to be considered is the following: Given the dynamical system $\frac{d\bar{x}}{dt} = \bar{f}(x,u)$ with $\bar{x}(t_o) = \bar{x}^o$ and a point $\bar{x}^1 \epsilon V$, find an admissible control $\bar{u} \epsilon U$ (if any exist) which transfers $\bar{x}^o$ to $\bar{x}^1$ such that F takes on the least value. In general, necessary conditions for the solution of this problem are given by the Pontryagin maximum principle (ref. 4).

In order to formulate the maximum principle, define $x_o(t)$ such that $\dot{x}_o(t) = f_o(\bar{x},\bar{u})$ $\left(x_o(t_o) = 0\right)$. $\left(\text{Note that } x_o(t_f) = F.\right)$ In addition to the system

$$\frac{dx_i}{dt} = f_i(\bar{x},\bar{u}) \qquad (i = 0, 1, 2, \ldots n) \qquad (1a)$$

consider another system of equations

$$\frac{d\psi_i}{dt} = -\sum_{j=0}^{n} \frac{\partial f_j}{\partial x_i}(\bar{x},\bar{u})\psi_j \qquad (i = 0, 1, 2, \ldots n) \qquad (1b)$$

in the auxiliary variables $\psi_o, \psi_1, \ldots \psi_n$ and define

$$\underset{\sim}{H} = \sum_{j=0}^{n} \psi_j f_j(\bar{x},\bar{u})$$

For fixed values of x_i and ψ_i $(i = 0, 1, \ldots n)$, $\underset{\sim}{H}$ becomes a function of $\bar{u} \epsilon U$.

<u>The Pontryagin maximum principle</u>.- Let $\bar{u}(t)$ $\left(t_o \leqq t \leqq t_f\right)$ be an admissible control which transfers $\bar{x}^o$ to $\bar{x}^1$ by equation (1a) such that $x_o(t_f)$ is minimized. For this case, it is necessary that there exist a nonzero continuous vector $\left[\psi_o(t), \psi_1(t), \ldots \psi_n(t)\right]'$ satisfying equation (1b) such that:

(1) The control $\bar{u}(t) \epsilon \bar{U}$ maximizes $\underset{\sim}{H}$ for fixed x_i and ψ_i $(i = 0, 1, \ldots n)$ at the point $\bar{u}(t)$; that is

$$\underset{\sim}{H}\left[x_i(t),\psi_i(t),u_j(t)\right] = \max_{\bar{u}\in U} \underset{\sim}{H} = \max_{\bar{u}\in U}\left[\sum_{k=0}^{n} \psi_k f_k(\bar{x},\bar{u})\right] = \underset{\sim}{M}\left[\psi_i(t),x_i(t)\right]$$

$$(i = 0,\ 1,\ \ldots\ n;\ j = 1,\ 2,\ \ldots\ r)$$

Then, $\underset{\sim}{M}\left[\psi_i(t),x_i(t)\right]$ represents the maximum value of $\underset{\sim}{H}$ attained by substituting $\bar{u} = \bar{u}(t)$ into $\underset{\sim}{H}$.

(2) For all $t\epsilon\left[t_o,t_f\right]$, $\psi_o(t) = \text{Constant} \leqq 0$ and $\underset{\sim}{M}\left[x_i(t),\psi_i(t)\right] = \text{Constant} = 0$.

The statement of the Pontryagin maximum principle is for an autonomous dynamic system with $\bar{x}^o$ and $\bar{x}^1$ given and t_f undetermined. Extensions of this theorem to autonomous systems with t_f fixed and to nonautonomous systems with t_f free and fixed are given in reference 4. Cases in which some $\bar{x}^o$ and $\bar{x}^1$ are unknown involve another facet of the theory known as the transversality condition (ref. 4). For illustrative purposes, consider the following example:

EXAMPLE:

Let the dynamical system be characterized by

$$\frac{d\bar{x}}{dt} = A\bar{x} + D\bar{u}$$

where $\bar{x}$ is an n-dimensional column vector, $\bar{u}$ is an r-dimensional column vector, and A and D are $(n \times n)$ and $(n \times r)$ matrices, respectively. Constrain the controls so that

$$\left|u_i\right| \leqq 1 \qquad (i = 1,\ 2,\ \ldots\ r)$$

Find $u_i(t)$ such that the dynamical system is transferred from $\bar{x}^o$ to $\bar{x}^1$ in minimal time; that is, $x_o(t_f) = t_f$ or $f_o(\bar{x},\bar{u}) = 1$.

From the Pontryagin maximum principle, the optimal control $\bar{u}(t)$ maximizes

$$\underset{\sim}{H} = \psi_o + \bar{\psi} \cdot \frac{d\bar{x}}{dt} = \psi_o + \bar{\psi}'A\bar{x} + \bar{\psi}'D\bar{u}$$

where $\psi_o \leqq 0$, $\bar{\psi}$ is the n-dimensional column vector $\bar{\psi} = \text{col}(\psi_1,\ \ldots\ \psi_n)$, and the prime denotes transpose.

Obviously, $\underset{\sim}{H}$ is maximized for $\bar{u}(t) = \text{sgn}\left[\bar{\psi}'D\right]'$ with $t_o \leqq t \leqq t_f$ where

$$\text{sgn } a = \begin{cases} 1 & (a > 0) \\ -1 & (a < 0) \\ \text{Undefined} & (a = 0) \end{cases}$$

The system

$$\frac{d\bar{x}}{dt}(t) = A\bar{x}(t) + D\ \text{sgn}\left[\bar{\psi}'D\right]'$$

$$\frac{d\bar{\psi}}{dt}(t) = -A'\bar{\psi}(t)$$

with the boundary conditions $\bar{x}(t_o) = \bar{x}^o$ and $\bar{x}(t_f) = \bar{x}^1$ now results. From condition (2) of the Pontryagin maximum principle

$$\bar{\psi}'(t)\left\{A\bar{x}(t) + D\,\mathrm{sgn}\left[\bar{\psi}'(t)D\right]'\right\} + \psi_o = 0$$

Hence, there exist (2n + 1) conditions for determining the variables $\bar{x}(t)$, $\bar{\psi}(t)$, and t_f.

Note that the form of the optimal control $\bar{u}(t)$ follows readily from the maximization of $\underset{\sim}{H}$. This feature is desirable. However, the optimal control is governed by $\bar{\psi}(t)$ and the initial conditions $\bar{\psi}(t_o)$ are not given. Thus, there exists a nonlinear two-point boundary-value problem which can be stated in the following form: Determine the (n + 2) unknown parameters $\psi_o \leqq 0$, $\bar{\psi}(t_o)$, and t_f such that at t_f, the (n + 1) terminal conditions $\bar{x}(t_f) = \bar{x}^1$ and $\underset{\sim}{M}\left[\bar{x}(t_f),\bar{\psi}(t_f)\right] = 0$ are met where $\bar{x}(t)$ and $\bar{\psi}(t)$ satisfy

$$\frac{d\bar{x}}{dt}(t) = A\bar{x}(t) + D\,\mathrm{sgn}\left[\psi'D\right]' \qquad \left(\bar{x}(t_o) = \bar{x}^o\right)$$

$$\frac{d\bar{x}}{dt}(t) = -A'\bar{\psi}(t)$$

Because ψ_o must be a constant greater than or equal to zero, the boundary-value problem can be separated into two cases. Both cases involve (n + 1) unknown parameters to be found so as to meet (n + 1) boundary conditions. In one case, ψ_o is set equal to zero; in the other case, ψ_o is chosen as some negative constant.

Such boundary-value problems typically result from the maximum principle and other indirect theories and are characteristic of their main difficulties. Generally, because $x_o(t)$ is completely specified by

$$\dot{x}_o = f_o(\bar{x},\bar{u}) \qquad \left(x_o(t_o) = 0\right)$$

and $\bar{x}(t)$ and $\bar{u}(t)$ are determined, x_o can be eliminated from the boundary-value problem. The corresponding auxiliary variable ψ_o can be eliminated by separating the problem into two cases as in the foregoing example. Thus, any two-point boundary-value problem originating from the maximum principle can be made to involve only the differential equations for x_i and ψ_i $(i = 1, 2, \ldots n)$.

A Particular Boundary-Value Problem

The purpose of this report is to present a successive approximation procedure for attacking a class of two-point boundary-value problems which frequently occurs in indirect optimization theory. The particular class of boundary-value problems to be considered and the mathematical assumptions concerning it are now presented.

The general result of applying an indirect method such as the Pontryagin maximum principle is a set of necessary conditions which can be arranged as 2n differential equations with mixed-boundary conditions. With the differential system defined over $[t_o,t_f]$, the boundary-value problem to be considered is one in which t_o is known, the optimal control $\bar{u}(t)$ is piecewise continuous, and there are $(m \leqq 2n)$ parameters represented by the column vector $\bar{\alpha} = (\alpha_1, \ldots \alpha_m)'$ to be chosen such that m terminal conditions are met. The parameters are some or all of the initial values of the differential equations and possibly the duration $t_f - t_o$. By writing $\bar{x}$ and $\bar{\psi}$ as $\bar{x}(\bar{\alpha},t)$ and $\bar{\psi}(\bar{\alpha},t)$, respectively, to indicate their dependence on $\bar{\alpha}$, the terminal conditions can be represented as

$$e_i\left[\bar{x}(\bar{\alpha},t_f),\bar{\psi}(\bar{\alpha},t_f),\bar{\alpha},t_f\right] = 0 \qquad (i = 1, 2, \ldots m)$$

The 2n differential equations can be written in the form

$$\left.\begin{aligned}
\frac{d}{dt}\bar{x}(\bar{\alpha},t) &= \bar{f}\left[\bar{x}(\bar{\alpha},t),\bar{u},\bar{\alpha},t\right] \\
\frac{d}{dt}\bar{\psi}(\bar{\alpha},t) &= -\left(\frac{\partial \bar{f}}{\partial \bar{x}}\right)'\left[\bar{x}(\bar{\alpha},t),\bar{u},\bar{\alpha},t\right]\bar{\psi}(\bar{\alpha},t) = \bar{g}\left[\bar{x}(\bar{\alpha},t),\bar{\psi}(\bar{\alpha},t),\bar{u},\bar{\alpha},t\right]
\end{aligned}\right\} \tag{2}$$

Assume that the optimal-control functions take the form

$$\bar{u} = \bar{u}\left\{\bar{x}(\bar{\alpha},t),\bar{\psi}(\bar{\alpha},t),\text{sgn}\ \bar{\rho}\left[\bar{x}(\bar{\alpha},t),\bar{\psi}(\bar{\alpha},t),\bar{\alpha},t\right],\bar{\alpha},t\right\} \tag{3}$$

where $\text{sgn}\ \bar{\rho}$ is an l-dimensional column vector with element $\text{sgn}\ \rho_i$. The $\rho_i(t)$ $(i = 1, 2, \ldots l)$ are continuous functions of t for given $\bar{x}(\bar{\alpha},t)$, $\bar{\psi}(\bar{\alpha},t)$, and $\bar{\alpha}$. Then, $\text{sgn}\ \rho_i(t)$ is defined as

$$\text{sgn}\ \rho_i(t) = \left\{\begin{array}{ll}
1 & (\rho_i(t) > 0) \\
-1 & (\rho_i(t) < 0) \\
\text{Undefined} & (\rho_i(t) = 0) \\
\qquad\text{except} & \\
\lim\limits_{t\to t_o^+} \text{sgn}\ \rho_i(t) & (\rho_i(t_o) = 0) \\
\lim\limits_{t\to t_f^-} \text{sgn}\ \rho_i(t) & (\rho_i(t_f) = 0)
\end{array}\right\} \tag{4}$$

The functions $\bar{u}$, $\bar{f}$, and $\bar{g}$ are to be continuous in $\bar{x}$, $\bar{\psi}$, $\bar{u}$, $\text{sgn}\ \bar{\rho}$, and $\bar{\alpha}$ and piecewise continuous in t with points of discontinuity occurring at those values of t for which $\rho_i(t) = 0$ $(i = 1, 2, \ldots l)$.

Assume that:

(1) For given $\bar{x}(t)$, $\bar{\psi}(t)$, and $\bar{\alpha}$, the zeros of $\rho_i(t)$ are finite in number where $i = 1, 2, \ldots l$.

(2) These zeros vary continuously with $\bar{\alpha}$.

(3) If t^* is a zero of $\rho_i(\bar{\alpha},t)$ for given $\bar{x}(\bar{\alpha},t)$ and $\bar{\psi}(\bar{\alpha},t)$, then $\rho_i(\bar{\alpha},t)$ is assumed to be continuously differentiable in t at t^* of order equal to the first nonvanishing derivative from the left of $\rho_i(\bar{\alpha},t)$ at t^*; that is, if p is the smallest positive integer such that $\rho_i^{(p)}(\bar{\alpha},t^*-) \neq 0$, then $\rho_i^{(p)}(\bar{\alpha},t^*-) = \rho_i^{(p)}(\bar{\alpha},t^*)$. Also, in such a case, $\frac{\partial}{\partial \alpha_j} \rho_i(\bar{\alpha},t)$ $(i = 1, 2, \ldots l;\ j = 1, 2, \ldots m)$ is assumed continuously differentiable in t at t^* of the order $p - 1$.

(4) The matrices $\frac{\partial f_i}{\partial \alpha_k}$ and $\frac{\partial g_i}{\partial \alpha_k}$ $(i = 1, 2, \ldots n;\ k = 1, 2, \ldots m)$ are bounded and continuous with finite interior limits as the boundaries are approached on the set $J(t_f) = [t_o,t_f] - [\text{The set of switching points of } \rho_j(t)\ (j = 1, 2, \ldots l)]$. Switching points are particular zeros of $\rho_j(t)$ as discussed subsequently.

An iterative procedure is now developed to improve upon an assumed value of $\bar{\alpha}$ in such a way that, when equations (2) with $\bar{u}$ given by equation (3) is satisfied, $e_i = 0$ $(i = 1, \ldots m)$ results.

THEORETICAL DEVELOPMENT OF CORRECTION TECHNIQUE

Iterative Logic

Given a value of $\bar{\alpha}$, equation (2) can be integrated and a set of values of $e_i\left(\bar{x}(\bar{\alpha},t_f),\bar{\psi}(\bar{\alpha},t_f),\bar{\alpha},t_f\right)$ $(i = 1, 2, \ldots m)$ can be computed. In order to simplify the notation, let these values of e_i be represented by $e_i(\bar{\alpha},t_f)$.

Let $\bar{e}(\bar{\alpha},t_f)$ be a column vector with elements $e_i(\bar{\alpha},t_f)$ $(i = 1, 2, \ldots m)$. Define a function $E(\bar{\alpha},t_f)$ as

$$E(\bar{\alpha},t_f) = \frac{\bar{e}(\bar{\alpha},t_f) \cdot B\bar{e}(\bar{\alpha},t_f)}{2} \tag{5}$$

where B is an $(m \times m)$ positive definite diagonal matrix. The scalar function $E(\bar{\alpha},t_f)$ can then be used as a measure of closeness to a solution because finding an $\bar{\alpha}$ for which $E(\bar{\alpha},t_f) = 0$ is equivalent to finding an $\bar{\alpha}$ for which $e_i(\bar{\alpha},t_f) = 0$ $(i = 1, 2, \ldots m)$. The elements of B are to be used as weighing coefficients.

Now assume a value of $\bar{\alpha}$; for example, $\bar{\alpha}^o$. Integrate equations (2) and evaluate $E(\bar{\alpha}^o,t_f)$. If $E(\bar{\alpha}^o,t_f) = 0$ for all practical purposes, then the boundary-value problem is solved; if not, let $\bar{\alpha}^o$ be changed by an amount $\delta\bar{\alpha}^o$. This change causes $\bar{e}(\bar{\alpha}^o,t_f)$ to become $\bar{e}(\bar{\alpha}^o + \delta\bar{\alpha}^o,t_f)$. Assume that

$$\frac{\partial\bar{e}}{\partial\bar{\alpha}}\left(\bar{\alpha}^o,t_f\right) \triangleq \left\{\frac{\partial e_i}{\partial\alpha_j}\left[\bar{x}\left(\bar{\alpha}^o,t_f\right),\bar{\psi}\left(\bar{\alpha}^o,t_f\right),\bar{\alpha},t_f\right]\right\}$$

exists for row $i = 1, 2, \ldots m$ and column $j = 1, 2, \ldots m$. Therefore

$$\Delta\bar{e}\left(\bar{\alpha}^o,t_f\right) \triangleq \bar{e}\left(\bar{\alpha}^o + \delta\bar{\alpha}^o,t_f\right) - \bar{e}\left(\bar{\alpha}^o,t_f\right) = \frac{d\bar{e}}{d\bar{\alpha}}\left(\bar{\alpha}^o,t_f\right)\delta\bar{\alpha}^o + o\left(\delta\bar{\alpha}^o\right) \tag{6}$$

where

$$\lim_{\left\|\delta\bar{\alpha}^o\right\|\to 0} \frac{\left\|o\left(\delta\bar{\alpha}^o\right)\right\|}{\left\|\delta\bar{\alpha}^o\right\|} = 0$$

The change $\Delta\bar{e}\left(\bar{\alpha}^o,t_f\right)$ causes a change $\Delta E\left(\bar{\alpha}^o,t_f\right)$ in $E\left(\bar{\alpha}^o,t_f\right)$ as given by

$$\Delta E\left(\bar{\alpha}^o,t_f\right) \triangleq E\left(\bar{\alpha}^o + \delta\bar{\alpha}^o,t_f\right) - E\left(\bar{\alpha}^o,t_f\right) = B\bar{e}\left(\bar{\alpha}^o,t_f\right) \cdot \Delta\bar{e}\left(\bar{\alpha}^o,t_f\right) + \frac{1}{2}\,\Delta\bar{e}\left(\bar{\alpha}^o,t_f\right) \cdot B\,\Delta\bar{e}\left(\bar{\alpha}^o,t_f\right)$$

Let $\tilde{\Delta}E\left(\bar{\alpha}^o,t_f\right)$ represent the value of $\Delta E\left(\bar{\alpha}^o,t_f\right)$ obtained by replacing $\Delta\bar{e}\left(\bar{\alpha}^o,t_f\right)$ by the first-order term in $\delta\bar{\alpha}^o$ of equation (6) so that

$$\tilde{\Delta}E\left(\bar{\alpha}^o,t_f\right) = \frac{\partial\bar{e}'}{\partial\bar{\alpha}}\left(\bar{\alpha}^o,t_f\right)B\bar{e}\left(\bar{\alpha}^o,t_f\right) \cdot \delta\bar{\alpha}^o + \frac{1}{2}\,\delta\bar{\alpha}^o \cdot \frac{\partial\bar{e}'}{\partial\bar{\alpha}}\left(\bar{\alpha}^o,t_f\right)B\,\frac{\partial\bar{e}}{\partial\bar{\alpha}}\left(\bar{\alpha}^o,t_f\right)\delta\bar{\alpha}^o \tag{7}$$

It will be assumed that $\left\|\delta\bar{\alpha}^o\right\|$ can be made so small that the behavior of $\Delta E\left(\bar{\alpha}^o,t_f\right)$ can be gathered from $\tilde{\Delta}E\left(\bar{\alpha}^o,t_f\right)$. In particular, $\left\|\delta\bar{\alpha}^o\right\|$ is so small that the algebraic sign of $\Delta E\left(\bar{\alpha}^o,t_f\right)$ is the same as that of $\tilde{\Delta}E\left(\bar{\alpha}^o,t_f\right)$. Analytically, this smallness on $\left\|\delta\bar{\alpha}^o\right\|$ can be represented by $\left\|\delta\bar{\alpha}^o\right\|^2 \leqq \nu^2$ $(\nu > 0)$.

In appendix A, lemmas 1 and 2 establish that unique solutions $\delta\bar{\alpha}^o(\nu)$ and $\lambda(\nu) < 0$ of the system

$$\left.\begin{aligned} &\left\|\delta\bar{\alpha}^o\right\|^2 = \nu^2 \\ &\delta\bar{\alpha}^o = -\left[\frac{\partial\bar{e}'}{\partial\bar{\alpha}}\left(\bar{\alpha}^o,t_f\right)B\,\frac{\partial\bar{e}}{\partial\bar{\alpha}}\left(\bar{\alpha}^o,t_f\right) - \lambda I\right]^{-1} \frac{\partial\bar{e}'}{\partial\bar{\alpha}}\left(\bar{\alpha}^o,t_f\right)B\bar{e}\left(\bar{\alpha}^o,t_f\right) \end{aligned}\right\} \tag{8}$$

exist if ν is sufficiently small (lemma 1). The solutions maximize the absolute value of $\tilde{\Delta}E\left(\bar{\alpha}^o,t_f\right)$ subject to the conditions $\left\|\delta\bar{\alpha}^o\right\| \leqq \nu^2$ and $\tilde{\Delta}E\left(\bar{\alpha}^o,t_f\right) < 0$ (lemma 2). Also, $\tilde{\Delta}E\left(\bar{\alpha}^o,t_f\right)$ is negative definite in $\delta\bar{\alpha}^o$ for $\delta\bar{\alpha}^o$ to satisfy equations (8) (lemma 3). Thus, given $\bar{\alpha}^o$ and ν where $\tilde{\Delta}E\left(\bar{\alpha}^o,t_f\right)$ approximates $\Delta E\left(\bar{\alpha}^o,t_f\right)$ and lemma 1 is satisfied, the replacement of $\bar{\alpha}^o$ by $\bar{\alpha}' = \bar{\alpha}^o + \delta\bar{\alpha}^o$, where $\delta\bar{\alpha}^o$ satisfies equations (8), yields $E\left(\bar{\alpha}',t_f\right) < E\left(\bar{\alpha}^o,t_f\right)$. This property follows from lemma 2. Repetitive use of this procedure generates a monotonic decreasing sequence of values of $E\left(\bar{\alpha}^i,t_f\right)$ with

$\bar{\alpha}^{i+1} = \bar{\alpha}^i + \delta\bar{\alpha}^i$ $(i = 0, 1, \ldots)$ and $\delta\bar{\alpha}^i$ satisfying equations (8). The values of ν for each i may not be the same. Because $E(\bar{\alpha}^i,t_f) \geqq 0$ $(i = 0, 1, \ldots)$, the sequence is bounded from below and therefore converges. The point of convergence is where $\tilde{\Delta}E(\bar{\alpha}^i,t_f)$ vanishes which, by lemma 3, occurs where $\delta\bar{\alpha}^i = \bar{0}$. Thus, the following iterative logic is established.

Theorem 1.- Given values of $\bar{\alpha}$ and ν, for example, $\bar{\alpha}^o$ and ν_o, respectively, for which $\tilde{\Delta}E(\bar{\alpha}^o,t_f)$ approximates $\Delta E(\bar{\alpha}^o,t_f)$ and lemma 1 holds, then the use of $\bar{\alpha}^{i+1} = \bar{\alpha}^i + \delta\bar{\alpha}^i$ $(i = 0, 1, \ldots)$ with $\delta\bar{\alpha}^i(\nu_i)$ and $|\lambda(\nu_i)|$ given by the simultaneous solution of

$$\left\|\delta\bar{\alpha}^i\right\|^2 = \nu_i^2 \tag{9}$$

and

$$\delta\bar{\alpha}^i = -\left[\frac{\partial\bar{e}'}{\partial\bar{\alpha}}(\bar{\alpha}^i,t_f)B\,\frac{\partial\bar{e}}{\partial\bar{\alpha}}(\bar{\alpha}^i,t_f) + \left|\lambda(\nu_i)\right|I\right]^{-1}\frac{\partial\bar{e}'}{\partial\bar{\alpha}}(\bar{\alpha}^i,t_f)B\bar{e}(\bar{\alpha}^i,t_f) \tag{10}$$

generates a monotonic decreasing sequence $E(\bar{\alpha}^i,t_f)$, if at each point of the sequence, values of ν_i can be found for which $\tilde{\Delta}E(\bar{\alpha}^i,t_f)$ approximates $\Delta E(\bar{\alpha}^i,t_f)$ and lemma 1 holds. The sequence $E(\bar{\alpha}^i,t_f)$ converges to a value of $E(\bar{\alpha},t_f)$ for which $\delta\bar{\alpha}$ of equation (10) vanishes.

The determination of values of ν_i for theorem 1 can be simplified by manipulating $|\lambda(\nu_i)|$ in equation (10) until $\tilde{\Delta}E(\bar{\alpha}^i,t_f)$ approximates $\Delta E(\bar{\alpha}^i,t_f)$ and then by computing ν_i from equation (9). A better approach is obtained by manipulating the $|\lambda(\nu_i)|$ in equation (10) until

$$\Delta E(\bar{\alpha}^i,t_f) = E(\bar{\alpha}^i + \delta\bar{\alpha}^i,t_f) - E(\bar{\alpha}^i,t_f) < 0$$

because theorem 1 may only establish sufficient (and not necessary) conditions that equations (9) and (10) generate a monotonic decreasing sequence $E(\bar{\alpha}^i,t_f)$.

Convergence

From equation (10), a limit point is reached when

$$\frac{\partial\bar{e}'}{\partial\bar{\alpha}}(\bar{\alpha}^i,t_f)B\bar{e}(\bar{\alpha}^i,t_f) = \bar{0}$$

The existence of the inverse of $\frac{\partial\bar{e}'}{\partial\bar{\alpha}}(\bar{\alpha}^i,t_f)$ determines whether $\bar{e}(\bar{\alpha}^i,t_f) = 0$. Obviously, the existence of $\frac{\partial\bar{e}}{\partial\bar{\alpha}}(\bar{\alpha}^i,t_f)^{-1}$ is sufficient but not necessary for $E(\bar{\alpha}^i,t_f)$ to vanish when $\frac{\partial\bar{e}'}{\partial\bar{\alpha}}(\bar{\alpha}^i,t_f)B\bar{e}(\bar{\alpha}^i,t_f)$ does. The vector $\frac{\partial\bar{e}'}{\partial\bar{\alpha}}(\bar{\alpha}^i,t_f)B\bar{e}(\bar{\alpha},t_f)$ can be recognized as $\frac{\partial E'}{\partial\bar{\alpha}}(\bar{\alpha},t_f)$ so that the arrival at $\frac{\partial\bar{e}'}{\partial\bar{\alpha}}(\bar{\alpha},t_f)B\bar{e}(\bar{\alpha},t_f) = \bar{0}$ means that an extremal of $E(\bar{\alpha},t_f)$ in the hyperspace $\bar{\alpha}, E(\bar{\alpha}, t_f)$ has been reached. The possibility of $\bar{e}(\bar{\alpha}, t_f) \neq \bar{0}$ at such a point always exists unless $E(\bar{\alpha}, t_f)$ is strictly convex in $\bar{\alpha}$; that is, the Hessian matrix $\frac{\partial^2 E}{\partial \bar{\alpha}^2}(\bar{\alpha}, t_f)$ is positive definite. Because

$$E(\bar{\alpha}, t_f) = \sum_{i=1}^{m} \frac{b_i e_i^2(\bar{\alpha}, t_f)}{2}$$

then

$$\frac{\partial^2 E}{\partial \bar{\alpha}^2}(\bar{\alpha}, t_f) = \sum_{i=1}^{m} b_i \left[\frac{\partial e_i'}{\partial \bar{\alpha}}(\bar{\alpha}, t_f) \frac{\partial e_i}{\partial \bar{\alpha}}(\bar{\alpha}, t_f) + e_i(\bar{\alpha}, t_f) \frac{\partial^2 e_i}{\partial \bar{\alpha}^2}(\bar{\alpha}, t_f)\right]$$

In general, it is not analytically evident that $\frac{\partial^2 E}{\partial \bar{\alpha}^2}(\bar{\alpha}, t_f)$ need be positive definite. Also, the computations necessary to evaluate $\partial^2 E / \partial \bar{\alpha}^2$ numerically and to test for positive definiteness would be excessive. It is therefore recommended that the procedure be used without attempting to examine the definiteness condition. However, note that when $\left\|\bar{e}(\bar{\alpha}, t_f)\right\|$ is so small that

$$\frac{\partial^2 E}{\partial \bar{\alpha}^2}(\bar{\alpha}, t_f) \cong \sum_{i=1}^{m} b_i \frac{\partial e_i'}{\partial \bar{\alpha}}(\bar{\alpha}, t_f) \frac{\partial e_i}{\partial \bar{\alpha}}(\bar{\alpha}, t_f)$$

then

$$\bar{\xi} \cdot \frac{\partial^2 E}{\partial \bar{\alpha}^2}(\bar{\alpha}, t_f) \bar{\xi} = \sum_{i=1}^{m} b_i \left[\frac{\partial e_i}{\partial \bar{\alpha}}(\bar{\alpha}, t_f) \bar{\xi}\right]^2 \geqq 0$$

when $\bar{\xi}$ is an arbitrary column vector; that is, $E(\bar{\alpha}, t_f)$ is locally convex when $\left\|\bar{e}(\bar{\alpha}, t_f)\right\|$ is small.

A procedure which relies on the numerical inversion of even a theoretically nonsingular matrix may incur stability problems. The matrix may be numerically near singularity in the sense that a substantial loss of significant figures results from the inversion process. The algorithm has the advantage that the nonsingularity of $\frac{\partial \bar{e}'}{\partial \bar{\alpha}}(\bar{\alpha}, t_f) B \frac{\partial \bar{e}}{\partial \bar{\alpha}}(\bar{\alpha}, t_f) + |\lambda| I$ can be controlled by manipulation of $|\lambda|$. Even though $\frac{\partial \bar{e}}{\partial \bar{\alpha}}(\bar{\alpha}, t_f)$ is singular, $\frac{\partial \bar{e}'}{\partial \bar{\alpha}}(\bar{\alpha}, t_f) B \frac{\partial \bar{e}}{\partial \bar{\alpha}}(\bar{\alpha}, t_f) + |\lambda| I$ can be made strongly nonsingular by increasing $|\lambda|$; thus numerical stability is provided by the iteration process.

Relation to Gradient and Newton-Raphson Processes

An important feature of the algorithm is observed in the limit as $|\lambda|$ either increases or decreases. As $|\lambda|$ increases, $\left[\frac{\partial \bar{e}'}{\partial \bar{\alpha}}(\bar{\alpha},t_f)B\frac{\partial \bar{e}}{\partial \bar{\alpha}} + |\lambda|I\right]^{-1}$ approaches $\frac{1}{|\lambda|}I$, and from equation (10)

$$\delta\bar{\alpha} \to -\frac{1}{|\lambda|}\frac{\partial \bar{e}'}{\partial \bar{\alpha}}(\bar{\alpha},t_f)B\bar{e}(\bar{\alpha},t_f) = -\frac{1}{|\lambda|}\frac{\partial E'}{\partial \bar{\alpha}}(\bar{\alpha},t_f) = \delta\bar{\alpha}_G \tag{11}$$

Equation (11) is simply the well-known method of gradients (ref. 7); that is, $\delta\bar{\alpha}$ is chosen to lie in the direction of the negative gradient of $E(\bar{\alpha},t_f)$ with respect to $\bar{\alpha}$ at $(\bar{\alpha},t_f)$. If $\left[\frac{\partial \bar{e}}{\partial \bar{\alpha}}(\bar{\alpha},t_f)\right]^{-1}$ exists, then as $|\lambda|$ decreases

$$\delta\bar{\alpha} \to -\left[\frac{\partial \bar{e}}{\partial \bar{\alpha}}(\bar{\alpha},t_f)\right]^{-1}\bar{e}(\bar{\alpha},t_f) = \delta\bar{\alpha}_{NR}$$

which can be recognized as a Newton-Raphson iteration procedure. For arbitrary λ, the process designated by equation (10) represents a combination of the gradient and Newton-Raphson techniques.

The result given by equation (10) is similar to that obtained in reference 8. Reference 8 shows that the angle between the vector $\delta\bar{\alpha}$ of equation (10) and the gradient vector of equation (11) is a continuous monotonic decreasing function of $|\lambda|$ such that as $|\lambda| \to \infty$, the angle approaches zero. Because the gradient vector of equation (11) is independent of $|\lambda|$, it follows that $\delta\bar{\alpha}$ of equation (10) rotates toward the $\delta\bar{\alpha}$ of equation (11) as $|\lambda| \to \infty$. The direction of the correction vector of equation (10) then lies between the directions given by the gradient and Newton-Raphson procedures; that is, $\delta\bar{\alpha}$ given by equation (10) is between $\delta\bar{\alpha}_G$ and $\delta\bar{\alpha}_{NR}$ in the sense that $\delta\bar{\alpha}$ always makes a smaller angle with the gradient direction of equation (11) than does $\delta\bar{\alpha}_{NR}$.

The Matrix $\frac{\partial \bar{e}}{\partial \bar{\alpha}}(\bar{\alpha},t_f)$

The form of the matrix $\frac{\partial \bar{e}}{\partial \bar{\alpha}}(\bar{\alpha},t_f)$ depends upon the nature of the final time t_f in the boundary-value problem. These forms can be divided into the following cases:

Case 1, where t_f is a constant.- Because $\bar{e}\left[\bar{x}(\bar{\alpha},t_f),\bar{\psi}(\bar{\alpha},t_f),\bar{\alpha},t_f\right]$ may depend implicitly, as well as explicitly, upon $\bar{\alpha}$ through the presence of $\bar{x}(\bar{\alpha},t_f)$ and $\bar{\psi}(\bar{\alpha},t_f)$, then

$$\frac{\partial e_j}{\partial \alpha_k}\left[\bar{x}(\bar{\alpha},t_f),\bar{\psi}(\bar{\alpha},t_f),\bar{\alpha},t_f\right] = \sum_{i=1}^{n} \frac{\partial e_j}{\partial x_i}(\bar{x},\bar{\psi},\bar{\alpha},t_f)\frac{\partial x_i}{\partial \alpha_k}(\bar{\alpha},t_f)$$

$$+ \sum_{i=1}^{n} \frac{\partial e_j}{\partial \psi_i}(\bar{x},\bar{\psi},\bar{\alpha},t_f)\frac{\partial \psi_i}{\partial \alpha_k}(\bar{\alpha},t_f) + \frac{\partial e_j}{\partial \alpha_k}(\bar{x},\bar{\psi},\bar{\alpha},t_f)$$

$$(j = 1, 2, \ldots m;\ k = 1, 2, \ldots m) \quad (12)$$

Case 2, where t_f is unspecified.- If the final time t_f is unspecified, it can be treated as a quantity which can be initially guessed and then corrected to meet the terminal conditions.

When the final time is unspecified, an (m - 1) dimensional parameter vector $\bar{\alpha}$ and t_f must be determined to meet m terminal conditions. Adding an mth column $\frac{\partial e_j}{\partial t_f}\left[\bar{x}(\bar{\alpha},t_f),\bar{\psi}(\bar{\alpha},t_f),\bar{\alpha},t_f\right]$ to the (m × m - 1) matrix $\frac{\partial \bar{e}}{\partial \bar{\alpha}}(\bar{\alpha},t_f)$ yields an (m × m) matrix which, when used in equation (10), generates a correction vector where the first (m - 1) elements represent $\delta\bar{\alpha}$ and the mth element represents δt_f. By this method, a correction to t_f can be computed which is influenced by the other $\delta\bar{\alpha}$. In the newly formed (m × m) matrix, the first (m - 1) columns have elements given by equation (12). The last column is given by

$$\frac{\partial e_j}{\partial t_f}\left[\bar{x}(\bar{\alpha},t_f),\bar{\psi}(\bar{\alpha},t_f),\bar{\alpha},t_f\right] = \frac{de_j}{dt}\left[\bar{x}(\bar{\alpha},t),\bar{\psi}(\bar{\alpha},t),\bar{\alpha},t\right]\Big|_{t=t_f}$$

$$+ \sum_{i=1}^{n} \frac{\partial e_j}{\partial x_i}(\bar{x},\bar{\psi},\bar{\alpha},t_f)\frac{\partial x_i}{\partial t_f}(\bar{\alpha},t_f) + \sum_{i=1}^{n} \frac{\partial e_j}{\partial \psi_i}(\bar{x},\bar{\psi},\bar{\alpha},t_f)\frac{\partial \psi_i}{\partial t_f}(\bar{\alpha},t_f)$$

$$(j = 1, 2, \ldots m) \quad (13)$$

The last two terms of equation (13) regard the final time as being fixed and take into account any changes in $\bar{x}(\bar{\alpha},t_f)$ and $\bar{\psi}(\bar{\alpha},t_f)$ as a result of a change in t_f at t_o. Thus, $\frac{\partial \bar{x}}{\partial t_f}(\bar{\alpha},t_f)$ and $\frac{\partial \bar{\psi}}{\partial t_f}(\bar{\alpha},t_f)$ satisfy the same type of equations as do $\frac{\partial \bar{x}}{\partial \alpha_k}(\bar{\alpha},t_f)$ and $\frac{\partial \bar{\psi}}{\partial \alpha_k}(\bar{\alpha},t_f)$ in equation (12).

Case 3, where t_f is implicitly specified.- A situation may occur in which the final time t_f is free to vary from iteration to iteration but is determined after a choice of $\bar{\alpha}$ is made through

$$t_f \in \left\{t, \varphi\left[\bar{x}(\bar{\alpha},t),\bar{\psi}(\bar{\alpha},t),\bar{\alpha},t\right] = 0\right\}$$

This case may be treated in one of the following manners once a t_f where $\varphi\left[\bar{x}(\bar{\alpha},t_f),\bar{\psi}(\bar{\alpha},t_f),\bar{\alpha},t_f\right] = 0$ has occurred:

(a) Choose $\left|\lambda(\bar{\nu}_i)\right|$ in equation (10) so large that the change in t_f from $\bar{\alpha}$ to $\bar{\alpha} + \delta\bar{\alpha}$ is small. Then, use the method in case 1 but iterate at the final time determined by φ.

(b) Adjoin $\varphi\left[\bar{x}(\bar{\alpha},t_f),\bar{\psi}(\bar{\alpha},t_f),\bar{\alpha},t_f\right]$ to $\bar{e}\left[\bar{x}(\bar{\alpha},t_f),\bar{\psi}(\bar{\alpha},t_f),\bar{\alpha},t_f\right]$ and t_f to $\bar{\alpha}$ and use the method as in case 2.

(c) Add to equation (12) the term

$$\frac{\partial e_j}{\partial t_f}(\bar{x},\bar{\psi},\bar{\alpha},t_f)\frac{\partial t_f}{\partial \alpha_k}(\bar{\alpha},t_f) \qquad (k = 1, 2, \ldots m)$$

where

$$\frac{\partial t_f}{\partial \alpha_k}(\bar{\alpha},t_f) = -\frac{\left\{\sum_{i=1}^{n}\left[\frac{\partial \varphi}{\partial x_i}(\bar{x},\bar{\psi},\bar{\alpha},t_f)\frac{\partial x_i}{\partial \alpha_k}(\bar{\alpha},t_f) + \frac{\partial \varphi}{\partial \psi_i}(\bar{x},\bar{\psi},\bar{\alpha},t_f)\frac{\partial \psi_i}{\partial \alpha_k}(\bar{\alpha},t_f)\right] + \frac{\partial \varphi}{\partial \alpha_k}(\bar{x},\bar{\psi},\bar{\alpha},t_f)\right\}}{\frac{d\varphi}{dt}(t_f)}$$

if

$$\frac{d\varphi}{dt}(t_f) \triangleq \frac{d}{dt}\,\varphi\left[\bar{x}(\bar{\alpha},t),\bar{\psi}(\bar{\alpha},t),\bar{\alpha},t\right]\Big|_{t=t_f} \neq 0$$

and take into account $\frac{\partial t_f}{\partial \alpha_k}(\bar{\alpha},t_f)$ in the computation of $\frac{\partial x_i}{\partial \alpha_k}(\bar{\alpha},t_f)$ and $\frac{\partial \psi_i}{\partial \alpha_k}(\bar{\alpha},t_f)$.

Cases 1 to 3 characterize $\frac{\partial \bar{e}}{\partial \bar{\alpha}}(\bar{\alpha},t_f)$. In order to complete the representations for $\frac{\partial \bar{e}}{\partial \bar{\alpha}}(\bar{\alpha},t_f)$, it is necessary to compute $\frac{\partial \bar{x}}{\partial \bar{\alpha}}(\bar{\alpha},t_f)$ and $\frac{\partial \bar{\psi}}{\partial \bar{\alpha}}(\bar{\alpha},t_f)$ for t_f treated as if it were fixed (cases 1, 2, 3(a), and 3(b)) and for case 3(c) where $\frac{\partial t_f}{\partial \bar{\alpha}_k}$ is actually considered. Before this computation is made, however, the zeros in $\left[t_o,t_f\right]$ of $\rho_q\left[\bar{x}(\bar{\alpha},t),\bar{\psi}(\bar{\alpha},t),\bar{\alpha},t\right]$ $(q = 1, 2, \ldots l)$ need to be discussed. Given $\bar{x}(\bar{\alpha},t)$ and $\bar{\psi}(\bar{\alpha},t)$, let $\hat{\rho}_q(\bar{\alpha},t) \triangleq \rho\left[\bar{x}(\bar{\alpha},t),\bar{\psi}(\bar{\alpha},t),\bar{\alpha},t\right]$, and given $\bar{\alpha}$, let $\tilde{\rho}_q(t) = \hat{\rho}_q(\bar{\alpha},t)$. As $\tilde{\rho}_q(t)$ passes to zero, $\operatorname{sgn}\tilde{\rho}_q(t)$ may not necessarily change value. If τ is a time point where $\tilde{\rho}_q(t) = 0$, fictitious points occur when the first nonvanishing derivative of $\tilde{\rho}_q(t)$ from the left at τ is of even order. These points occur where $\tilde{\rho}_q(t)$ has a local maximum or minimum (ref. 9). At such points, $\operatorname{sgn}\tilde{\rho}_q(\tau^-) = \operatorname{sgn}\tilde{\rho}_q(\tau^+)$. However, $\operatorname{sgn}\tilde{\rho}_q(t)$ does change value for $\tilde{\rho}_q(\tau) = 0$, and the first nonvanishing derivative from the left at τ is of odd order. These values of τ, denoted by t^*, are referred to as switching points of the switching function $\tilde{\rho}_q(t)$ to distinguish them from zeros of $\tilde{\rho}_q(t)$ where sgn $\tilde{\rho}_q(t)$ does not change sign. Let the set of switching points of $\tilde{\rho}_q(t)$ be denoted by $S_q(t^*)$ and $S(t^*)$ denote the set of all switching points of all $\tilde{\rho}_q(t)$ $(q = 1, 2, \ldots l)$. Because the total number of switching points is assumed to be finite, the elements of $S(t^*)$ can be ordered such that

$$S(t^*) = \left(t_1^*, t_2^*, \ldots t_v^*\right)$$

where $t_{i+1}^* > t_i^*$ for $i = 1, 2, \ldots v$.

Cases 1, 2, 3(a), 3(b).- In order to compute $\frac{\partial\bar{x}}{\partial\bar{\alpha}}\left(\bar{\alpha},t_f\right)$ and $\frac{\partial\bar{\psi}}{\partial\bar{\alpha}}\left(\bar{\alpha},t_f\right)$ when t_f is fixed, $\frac{\partial\bar{x}}{\partial\bar{\alpha}}\left(\bar{\alpha},t_f\right)$ is derived by assuming that $\frac{\partial\bar{\psi}}{\partial\bar{\alpha}}\left(\bar{\alpha},t_f\right)$ is known. The equation for $\frac{\partial\bar{\psi}}{\partial\bar{\alpha}}\left(\bar{\alpha},t_f\right)$ then follows from observation of the equation for $\frac{\partial\bar{x}}{\partial\bar{\alpha}}\left(\bar{\alpha},t_f\right)$.

When $\bar{x}(\bar{\alpha},t)$, $\psi(\bar{\alpha},t)$, and $\bar{\alpha}$ are given, the function $\bar{f}$ is a piecewise continuous function of t with points of discontinuity occurring only on $S(t^*)$. Thus, $\bar{x}(\bar{\alpha},t)$ is continuous in t and satisfies the integral equation

$$\bar{x}(\bar{\alpha},t) = \bar{x}\left(\bar{\alpha},t_o\right) + \int_{t_o}^{t} \bar{f}(\bar{x},\bar{u},\bar{\alpha},s)ds$$

When $\bar{x}(\bar{\alpha},t)$, $\bar{\psi}(\bar{\alpha},t)$, and $\bar{\alpha}$ are given, $\tilde{\bar{f}}(t) = \bar{f}(\bar{x},\bar{u},\bar{\alpha},t)$. Because $\tilde{\bar{f}}(t)$ is bounded on $S(t^*)$

$$\bar{x}\left(\bar{\alpha},t_f\right) = \bar{x}\left(\bar{\alpha},t_o\right) + \int_{J(t_f)} \bar{f}(\bar{x},\bar{u},\bar{\alpha},s)ds$$

$$= \lim_{\epsilon\to 0}\left[\int_{t_o}^{t_1^*-\epsilon} \tilde{\bar{f}}(s)ds + \sum_{j=1}^{v-1}\int_{t_j^*+\epsilon}^{t_{j+1}^*-\epsilon} \tilde{\bar{f}}(s)ds + \int_{t_v^*+\epsilon}^{t_f} \tilde{\bar{f}}(s)ds\right]$$

where

$$J(t_f) = \left[t_o, t_f\right] - S(t^*)$$

Let $\frac{\partial\hat{\bar{f}}}{\partial\bar{\alpha}}(\bar{\alpha},t) = \frac{\partial\bar{f}}{\partial\bar{\alpha}}(\bar{x},\bar{u},\bar{\alpha},t)$ for given $\bar{x}(\bar{\alpha},t)$ and $\bar{\psi}(\bar{\alpha},t)$, and $\bar{\alpha}$. Furthermore, because $\frac{\partial\hat{\bar{f}}}{\partial\bar{\alpha}}(\bar{\alpha},t)$ is bounded and continuous on $J\left(t_f\right)$

$$\frac{\partial \bar{x}}{\partial \bar{\alpha}}(\bar{\alpha},t_f) = \frac{\partial \bar{x}}{\partial \bar{\alpha}}(\bar{\alpha},t_f) + \sum_{j=1}^{v} \sum_{\tilde{\rho}_q(t_j^*)}^{1-l} \lim_{\epsilon \to 0}\left[\tilde{\bar{f}}\left(t_j^* - \epsilon\right)\frac{\partial t_j^*}{\partial \bar{\alpha}}\left(\bar{\alpha},t_j^* - \epsilon\right)\right.$$

$$\left. - \tilde{\bar{f}}(t^* + \epsilon)\ \frac{\partial t_j^*}{\partial \bar{\alpha}}(\bar{\alpha},t^* + \epsilon)\right] + \int_{J(t_f)} \frac{\partial \hat{\bar{f}}}{\partial \bar{\alpha}}(\bar{\alpha},s)ds$$

where the symbol $\sum\limits_{\tilde{\rho}_q(t_j^*)}^{1-l}$ means the sum over all $\tilde{\rho}_q$ $(q = 1, 2, \ldots l)$ having t_j^* as a common switching point.

For arbitrary $t\epsilon\left[t_o,t_f\right]$, $\frac{\partial \bar{x}}{\partial \bar{\alpha}}(\bar{\alpha},t)$ becomes the solution of the integral equation

$$\frac{\partial \bar{x}}{\partial \bar{\alpha}}(\bar{\alpha},t) = \frac{\partial \bar{x}}{\partial \bar{\alpha}}(\bar{\alpha},t_o) + \sum_{j=1}^{v}\left\{\sum_{\tilde{\rho}_q(t_j^*)}^{1-l}\left[\tilde{\bar{f}}\left(t_j^{*-}\right)\frac{\partial t_j^*}{\partial \bar{\alpha}}\left(\bar{\alpha},t_j^{*-}\right)\right.\right.$$

$$\left.\left. - \tilde{\bar{f}}\left(t_j^{*+}\right)\frac{\partial t_j^*}{\partial \bar{\alpha}}\left(\bar{\alpha},t_j^{*+}\right)\right]\right\} H\left(t - t_j^*\right) + \int_{J(t)} \frac{\partial \tilde{\bar{f}}}{\partial \bar{\alpha}}(\bar{\alpha},s)ds$$

where

$$H\left(t - t_j\right) = \begin{cases} 0 & \left(t < t_j^*\right) \\ 1 & \left(t > t_j^*\right) \end{cases}$$

and

$$J(t) = \left[t_o,t\right] - S(t^*)$$

For a switching point t^* of $\hat{\rho}_q(\bar{\alpha},t)$ and a parameter vector $\bar{\alpha}$, an infinitesimal change $d\bar{\alpha}$ in $\bar{\alpha}$ causes a change $d\hat{\rho}_q(\bar{\alpha},t^*)$ away from zero. If $\hat{\rho}_q(\bar{\alpha} + d\bar{\alpha},t^* + dt^*)$ is also to be zero, then

$$dt^* = \sum_{k=1}^{m} \frac{\dfrac{\partial \hat{\rho}_q}{\partial \alpha_k}(\bar{\alpha},t^*)d\alpha_k}{\dfrac{d\hat{\rho}_q}{dt}(\bar{\alpha},t^*)}$$

where

$$\frac{\partial\hat{\rho}_q}{\partial\alpha_k}(\bar{\alpha},t^*) = \frac{\partial\rho_q}{\partial\alpha_k}\left[\bar{x}(\bar{\alpha},t),\bar{\psi}(\bar{\alpha},t),\bar{\alpha},t\right]$$

$$= \sum_{i=1}^{n}\left[\frac{\partial\rho_q}{\partial x_i}(\bar{x},\bar{\psi},\bar{\alpha},t^*)\frac{\partial x_i}{\partial\alpha_k}(\bar{\alpha},t^*) + \frac{\partial\rho_q}{\partial\psi_i}(\bar{x},\bar{\psi},\bar{\alpha},t^*)\frac{\partial\psi_i}{\partial\alpha_k}(\bar{\alpha},t^*)\right] + \frac{\partial\rho_q}{\partial\alpha_k}(\bar{x},\bar{\psi},\bar{\alpha},t^*)$$

and

$$\frac{d}{dt}\,\hat{\rho}_q(\bar{\alpha},t^*) = \frac{d}{dt}\,\rho_q\left[\bar{x}(\bar{\alpha},t),\bar{\psi}(\bar{\alpha},t),\bar{\alpha},t\right]\Big|_{t=t^*}$$

The condition that the zeros of $\hat{\rho}_q(\bar{\alpha},t)$ in t change continuously with α_k implies that $\dfrac{\frac{\partial}{\partial\alpha_k}\hat{\rho}_q(\bar{\alpha},t)}{\frac{d}{dt}\tilde{\rho}_q(\bar{\alpha},t)}$ $(q = 1, 2, \ldots l;\ k = 1, 2, \ldots m)$ are continuous functions of t at t^* for given $\bar{\alpha}$. Thus, in order to preserve this continuity, $\frac{\partial\hat{\rho}_q}{\partial\alpha_k}(\bar{\alpha},t^*)$ must vanish at any switching point where $\frac{d}{dt}\,\hat{\rho}_q(\bar{\alpha},t^*) = 0$. In general, $\dfrac{\frac{\partial\hat{\rho}_q}{\partial\alpha_k}(\bar{\alpha},t^*)}{\frac{d\hat{\rho}_q}{dt}(\bar{\alpha},t^*)}$ can be replaced by $\dfrac{\frac{d^{p-1}}{dt^{p-1}}\left[\frac{\partial\hat{\rho}_q}{\partial\alpha_k}(\bar{\alpha},t^*)\right]}{\frac{d^p\hat{\rho}_q}{dt^p}(\bar{\alpha},t)}$ where p is the order of the first nonvanishing derivative from the left of $\tilde{\rho}_q(t)$ at t^*. Because only switching points of $\tilde{\rho}_q(t)$ are used, p can be considered odd. The change in a switching point t^* with respect to a change in a parameter α_k for a switching function ρ_q is then given by

$$\frac{\partial t^*}{\partial\alpha_k}(\bar{\alpha},t^*) = -\frac{\dfrac{d^{p-1}}{dt^{p-1}}\left[\dfrac{\partial\hat{\rho}_q}{\partial\alpha_k}(\bar{\alpha},t^*)\right]}{\dfrac{d^p\hat{\rho}_q}{dt^p}(\bar{\alpha},t^*)} \tag{14}$$

which, by assumption, is continuous at t^*. By using equation (14) and by recognizing that $\operatorname{sgn}\tilde{\rho}_q(t)$ is fixed over the intervals between the switching points, $\frac{\partial\bar{x}}{\partial\bar{\alpha}}(\bar{\alpha},t)$ can be written as

$$\frac{\partial\bar{x}}{\partial\bar{\alpha}}(\bar{\alpha},t) = \frac{\partial\bar{x}}{\partial\bar{\alpha}}(\bar{\alpha},t_o) + \sum_{j=1}^{v}\left\{\sum_{\tilde{\rho}_q(t_j^*)}^{1-l}\left[\tilde{\bar{f}}(t_j^{*+}) - \tilde{\bar{f}}(t_j^{*-})\right]\frac{\frac{d^{p-1}}{dt^{p-1}}\frac{\partial\hat{\rho}_q}{\partial\bar{\alpha}}(\bar{\alpha},t^*)}{\frac{d^p\hat{\rho}_q}{dt^p}(\bar{\alpha},t_j^*)}\right\}H(t - t_j^*)$$

$$+ \int_{J(t)}\left[\left(\frac{\partial\bar{f}}{\partial\bar{x}} + \frac{\partial\bar{f}}{\partial\bar{u}}\frac{\partial\bar{u}}{\partial\bar{x}}\right)\frac{\partial\bar{x}}{\partial\bar{\alpha}} + \frac{\partial\bar{f}}{\partial\bar{u}}\frac{\partial\bar{u}}{\partial\bar{\psi}}\frac{\partial\bar{\psi}}{\partial\bar{\alpha}} + \frac{\partial\bar{f}}{\partial\bar{u}}\frac{\partial\bar{u}}{\partial\bar{\alpha}} + \frac{\partial\bar{f}}{\partial\bar{\alpha}}\right]ds \tag{15}$$

for given values of $\bar{x}(\bar{\alpha},t)$, $\bar{\psi}(\bar{\alpha},t)$, $\frac{\partial\bar{\psi}}{\partial\bar{\alpha}}(\bar{\alpha},t)$, and $\bar{\alpha}$.

Computationally, this integral equation can be solved in the following manner. Given $\bar{x}(\bar{\alpha},t)$, $\bar{\psi}(\bar{\alpha},t)$, $\frac{\partial\bar{\psi}}{\partial\bar{\alpha}}(\bar{\alpha},t)$, and $\bar{\alpha}$, integrate the differential equation

$$\frac{d}{dt}\left(\frac{\partial\bar{x}}{\partial\bar{\alpha}}\right) = \left(\frac{\partial\bar{f}}{\partial\bar{x}} + \frac{\partial\bar{f}}{\partial\bar{u}}\frac{\partial\bar{u}}{\partial\bar{x}}\right)\frac{\partial\bar{x}}{\partial\bar{\alpha}} + \frac{\partial\bar{f}}{\partial\bar{u}}\frac{\partial\bar{u}}{\partial\bar{\psi}}\frac{\partial\bar{\psi}}{\partial\bar{\alpha}} + \frac{\partial\bar{f}}{\partial\bar{u}}\frac{\partial\bar{u}}{\partial\bar{\alpha}} + \frac{\partial\bar{f}}{\partial\bar{\alpha}} \tag{16}$$

with $\frac{\partial\bar{x}}{\partial\bar{\alpha}}(t_o) = \frac{\partial\bar{x}}{\partial\bar{\alpha}}(\bar{\alpha},t_o)$ from t_o to t_1^*. Call this $\frac{\partial\bar{x}}{\partial\bar{\alpha}}(\bar{\alpha},t_1^{*-})$. Replace $\frac{\partial\bar{x}}{\partial\bar{\alpha}}(\bar{\alpha},t_1^{*-})$ by

$$\frac{\partial\bar{x}}{\partial\bar{\alpha}}(\bar{\alpha},t_1^{*-}) + \sum_{\tilde{\rho}_q(t_1^*)}^{1-l}\left[\tilde{\bar{f}}(t_1^{*+}) - \tilde{\bar{f}}(t_1^{*-})\right]\frac{\frac{d^{p-1}}{dt^{p-1}}\frac{\partial\hat{\rho}_q}{\partial\bar{\alpha}}(\bar{\alpha},t_1^*)}{\frac{d^p\hat{\rho}_q}{dt^p}(\bar{\alpha},t_1^*)}$$

and use this as the initial condition for the integration of equation (16) from t_1^{*+} to t_2^{*-}. Repeat the process until the desired t is obtained. The quantity $\tilde{\bar{f}}(t_j^{*+}) - \tilde{\bar{f}}(t_j^{*-})$ is simply the "jump" that $\tilde{\bar{f}}(t)$ takes in going from t_j^{*-} to t_j^{*+}.

Because $\bar{g}$ is subject to the same restrictions as $\bar{f}$, $\frac{\partial\bar{\psi}}{\partial\bar{\alpha}}(\bar{\alpha},t)$ satisfies the integral equation

$$\frac{\partial\bar{\psi}}{\partial\bar{\alpha}}(\bar{\alpha},t) = \frac{\partial\bar{\psi}}{\partial\bar{\alpha}}(\bar{\alpha},t_o) + \sum_{j=1}^{v}\left\{\sum_{\tilde{\rho}_q(t_j^*)}^{1-l}\left[\tilde{\bar{g}}(t_j^{*+}) - \tilde{\bar{g}}(t_j^{*-})\right]\frac{\frac{d^{p-1}}{dt^{p-1}}\frac{\partial\hat{\rho}_q}{\partial\bar{\alpha}}(\bar{\alpha},t_j^*)}{\frac{d^p}{dt^p}\hat{\rho}_q(\bar{\alpha},t^*)}\right\}H(t - t_j^*)$$

$$+ \int_{J(t)}\left[\left(\frac{\partial\bar{g}}{\partial\bar{x}} + \frac{\partial\bar{g}}{\partial\bar{u}}\frac{\partial\bar{u}}{\partial\bar{x}}\right)\frac{\partial\bar{x}}{\partial\bar{\alpha}} + \left(\frac{\partial\bar{g}}{\partial\bar{x}} + \frac{\partial\bar{g}}{\partial\bar{u}}\frac{\partial\bar{u}}{\partial\bar{\psi}}\right)\frac{\partial\bar{\psi}}{\partial\bar{\alpha}} + \frac{\partial\bar{g}}{\partial\bar{u}}\frac{\partial\bar{u}}{\partial\bar{\alpha}} + \frac{\partial\bar{g}}{\partial\bar{\alpha}}\right]ds \tag{17}$$

Computationally, equation (17) can be treated the same as equation (15). Given $\bar{\alpha}$ and the solution of equation (2), equations (15) and (17) yield a simultaneous set of matrix integral equations for the determination of $\frac{\partial \bar{x}}{\partial \bar{\alpha}}(\bar{\alpha},t_f)$ and $\frac{\partial \bar{\psi}}{\partial \bar{\alpha}}(\bar{\alpha},t_f)$ for use in cases 1, 2, 3(a), and 3(b). It can be noted that $\frac{\partial \bar{x}}{\partial \bar{\alpha}}(\bar{\alpha},t)$ and $\frac{\partial \bar{\psi}}{\partial \bar{\alpha}}(\bar{\alpha},t)$ are piecewise continuous functions over $[t_o,t_f]$ with discontinuities occurring on $S(t^*)$.

The initial conditions $\frac{\partial \psi_i}{\partial \alpha_k}(\bar{\alpha},t_o)$ and $\frac{\partial x_i}{\partial \alpha_k}(\bar{\alpha},t_o)$ $(i = 1, 2, \ldots n;$ $k = 1, 2, \ldots m)$ are to be determined from the nature of $\bar{\alpha}$ in a particular problem; for example, if $\alpha_1 = \psi_1(t_o)$

$$\frac{\partial \psi_i}{\partial \alpha_1}(t_o) = \begin{cases} 1 & (i = 1) \\ 0 & (i = 2, \ldots n) \end{cases}$$

and $\frac{\partial x_i}{\partial \alpha_j}(t_o) = 0$ $(i = 1, 2, \ldots n;\ j = 1, 2, \ldots n)$.

Case 3(c).- Let $\frac{\partial \bar{x}}{\partial \bar{\alpha}}(\bar{\alpha},t_f)$ and $\frac{\partial \bar{\psi}}{\partial \bar{\alpha}}(\bar{\alpha},t_f)$, given by equations (15) and (17), be denoted by $X(\bar{\alpha},t_f)$ and $\Psi(\bar{\alpha},t_f)$, respectively. In case 3(c), an extra term must be added to equations (15) and (17) so that they become

$$\frac{\partial \bar{x}}{\partial \bar{\alpha}}(\bar{\alpha},t_f) = X(\bar{\alpha},t_f) + \left[\frac{\partial \bar{x}}{\partial t_f}(\bar{\alpha},t_f) + \tilde{\bar{f}}(t_f)\right]\frac{\partial t_f}{\partial \bar{\alpha}}(\bar{\alpha},t_f)$$

and

$$\frac{\partial \bar{\psi}}{\partial \bar{\alpha}}(\bar{\alpha},t_f) = \Psi(\bar{\alpha},t_f) + \left[\frac{\partial \bar{\psi}}{\partial t_f}(\bar{\alpha},t_f) + \tilde{\bar{g}}(t_f)\right]\frac{\partial t_f}{\partial \bar{\alpha}}(\bar{\alpha},t_f)$$

respectively. The variables $\frac{\partial \bar{x}}{\partial t_f}(\bar{\alpha},t_f)$ and $\frac{\partial \bar{\psi}}{\partial t_f}(\bar{\alpha},t_f)$ are computed by treating t_f as a parameter and by using equations (15) and (17). In order to solve for $\frac{\partial \bar{x}}{\partial \bar{\alpha}}(\bar{\alpha},t_f)$ and $\frac{\partial \bar{\psi}}{\partial \bar{\alpha}}(\bar{\alpha},t_f)$, the linear system

$$\left\{\frac{d\varphi}{dt}(t_f)I + \left[\frac{\partial \bar{x}}{\partial t_f}(\bar{\alpha},t_f) + \tilde{\bar{f}}(t_f)\right]\frac{\partial \varphi}{\partial \bar{x}}(\bar{x},\bar{\psi},\bar{\alpha},t_f)\right\}\frac{\partial \bar{x}}{\partial \bar{\alpha}}(\bar{\alpha},t_f) + \left[\frac{\partial \bar{x}}{\partial t_f}(\bar{\alpha},t_f) + \tilde{\bar{f}}(t_f)\right]\frac{\partial \varphi}{\partial \bar{\psi}}(\bar{x},\bar{\psi},\bar{\alpha},t_f)\frac{\partial \bar{\psi}}{\partial \bar{\alpha}}(\bar{\alpha},t_f)$$

$$= \frac{d\varphi}{dt}(t_f)X(\bar{\alpha},t_f) - \left[\frac{\partial \bar{x}}{\partial t_f}(\bar{\alpha},t_f) + \tilde{\bar{f}}(t_f)\right]\frac{\partial \varphi}{\partial \bar{\alpha}}(\bar{x},\bar{\psi},\bar{\alpha},t_f)$$

and

$$\left[\frac{\partial\bar{\psi}}{\partial t_f}(\bar{\alpha},t_f) + \tilde{\bar{g}}(t_f)\right]\frac{\partial\varphi}{\partial\bar{x}}(\bar{x},\bar{\psi},\bar{\alpha},t_f)\frac{\partial\bar{x}}{\partial\bar{\alpha}}(\bar{\alpha},t_f) + \left\{\frac{d\varphi}{dt}(t_f)I + \left[\frac{\partial\bar{\psi}}{\partial t_f}(\bar{\alpha},t_f) + \tilde{\bar{g}}(t_f)\right]\frac{\partial\varphi}{\partial\bar{\psi}}(\bar{x},\bar{\psi},\bar{\alpha},t_f)\right\}\frac{\partial\bar{\psi}}{\partial\bar{\alpha}}(\bar{\alpha},t_f)$$

$$= \frac{d\varphi}{dt}(t_f)\Psi(\bar{\alpha},t_f) - \left[\frac{\partial\bar{\psi}}{\partial t_f}(\bar{\alpha},t_f) + \tilde{\bar{g}}(t_f)\right]\frac{\partial\varphi}{\partial\bar{\alpha}}(\bar{x},\bar{\psi},\bar{\alpha},t_f)$$

must be solved.

This system requires the inversion of the (2n × 2n) matrix

$$\begin{bmatrix} \frac{d\varphi}{dt}(t_f)I + \left[\frac{\partial\bar{x}}{\partial t_f}(\bar{\alpha},t_f) + \tilde{\bar{f}}(t_f)\right]\frac{\partial\varphi}{\partial\bar{x}}(\bar{x},\bar{\psi},\bar{\alpha},t_f) & \left[\frac{\partial\bar{x}}{\partial t_f}(\bar{\alpha},t_f) + \tilde{\bar{f}}(t_f)\right]\frac{\partial\varphi}{\partial\bar{\psi}}(\bar{x},\bar{\psi},\bar{\alpha},t_f) \\ \left[\frac{\partial\bar{\psi}}{\partial t_f}(\bar{\alpha},t_f) + \tilde{\bar{g}}(t_f)\right]\frac{\partial\varphi}{\partial\bar{x}}(\bar{x},\bar{\psi},\bar{\alpha},t_f) & \frac{d\varphi}{dt}(t_f)I + \left[\frac{\partial\bar{\psi}}{\partial t_f}(\bar{\alpha},t_f) + \tilde{\bar{g}}(t_f)\right]\frac{\partial\varphi}{\partial\bar{\psi}}(\bar{x},\bar{\psi},\bar{\alpha},t_f) \end{bmatrix} \quad (18)$$

Difficulty may arise when equation (18) is singular or if too many significant figures are lost during the inversion process.

Critical Review

The procedure presented provides several important advantages. Only a forward integration and a single matrix inversion must be performed to compute the correction vector given by equation (10). The matrix to be inverted is guaranteed to be nonsingular. The direction of the correction vector lies between the direction given by the gradient and the Newton-Raphson processes. An additional advantage is that a final-time correction can be made an integral part of the process provided that the final time is an unknown parameter; that is, corrections in the final time can be computed at each iteration as a component of the parameter correction vector. Also, integral equations are available for influence matrices that describe the effect of a change in the parameters on the terminal conditions.

The process also has some disadvantages. The fact that complete convergence can be obtained from an arbitrary choice of assumed parameters is not established. The technique proposed is basically a boundary-condition iteration scheme. Such schemes generally have sensitivity and convergence problems (ref. 10). The process does not generally eliminate such difficulties. Finally, the technique cannot be used in problems in which a singular control (ref. 11) might occur unless such an occurrence can be predicted.

EXAMPLE CALCULATION

Problem Statement

In order to exemplify the usefulness of the foregoing method, the two-point boundary-value problem arising from the application of the Pontryagin maximum principle (ref. 4) to determine fuel-optimal lunar-rendezvous trajectories is considered. The dynamic equations for rendezvous with a target in a circular orbit are developed in appendix B. The maximum principle is applied in appendix C. The result is a two-point boundary problem of the type considered.

Application of Algorithm

The foregoing policy of exhibiting all variables of a function is not continued because subsequent equations are quite lengthy and involved. For example, instead of writing $\rho\left(\bar{x},\bar{\psi},\bar{\alpha},t\right)$, ρ or $\rho(t)$ is written with $\rho\left(\bar{x},\bar{\psi},\bar{\alpha},t\right)$ implied by the defining equation.

From appendix C, the system of equations corresponding to equation (2) is

$$f_1 = \dot{x}_1 = x_2$$

$$f_2 = \dot{x}_2 = \frac{\nu\left[1 + \operatorname{sgn}\rho\right]\psi_2}{2\sqrt{\psi}\,x_7} - \frac{\Omega^2 R_S{}^3\left(x_1 + R_{S_x}\right)}{\sqrt{x}^{\,3}} + \Omega^2 R_{S_x} + \omega^2 x_1 + 2\omega x_4$$

$$f_3 = \dot{x}_3 = x_4$$

$$f_4 = \dot{x}_4 = \frac{\nu\left[1 + \operatorname{sgn}\rho\right]\psi_4}{2\sqrt{\psi}\,x} - \frac{\Omega^2 R_S{}^3\left(x_3 + R_{S_y}\right)}{\sqrt{x}^{\,3}} + \Omega^2 R_{S_y} - 2\omega x_2 + \omega^2 x_3$$

$$f_5 = \dot{x}_5 = x_6$$

$$f_6 = \dot{x}_6 = \frac{\nu\left[1 + \operatorname{sgn}\rho\right]\psi_6}{2\sqrt{\psi}\,x_7} - \frac{\Omega R_S\left(x_5 + R_{S_z}\right)}{\sqrt{x}^{\,3}} + \Omega^2 R_{S_z}$$

$$f_7 = \dot{x}_7 = -\frac{\nu\left[1 + \operatorname{sgn}\rho\right]}{2c}$$

$$g_1 = \dot{\psi}_1 = \frac{\Omega^2 R_s{}^3 \psi_2}{\sqrt{x}^3} - \frac{3\Omega^2 R_s{}^3 d}{\sqrt{x}^5}\left(x_1 + R_{s_x}\right) - \omega^2 \psi_2$$

$$g_2 = \dot{\psi}_2 = -\psi_1 + 2\omega\psi_4$$

$$g_3 = \dot{\psi}_3 = \frac{\Omega^2 R_s{}^3 \psi_4}{\sqrt{x}^3} - \frac{3\Omega^2 R_s{}^3 d}{\sqrt{x}^5}\left(x_3 + R_{s_y}\right) - \omega^2 \psi_4$$

$$g_4 = \dot{\psi}_4 = -2\omega\psi_2 - \psi_3$$

$$g_5 = \dot{\psi}_5 = \frac{\Omega^2 R_s{}^3 \psi_6}{\sqrt{x}^3} - \frac{3\Omega^2 R_s{}^3 d}{\sqrt{x}^5}\left(x_5 + R_{s_z}\right)$$

$$g_6 = \dot{\psi}_6 = -\psi_5$$

$$g_7 = \dot{\psi}_7 = \frac{\gamma}{2} \frac{\left[1 + \operatorname{sgn} \rho\right]\sqrt{\psi}}{x_7{}^2}$$

where

$$d = \psi_2\left(x_1 + R_{s_x}\right) + \psi_4\left(x_3 + R_{s_y}\right) + \psi_6\left(x_5 + R_{s_z}\right)$$

$$\sqrt{\psi} = \sqrt{\psi_2{}^2 + \psi_4{}^2 + \psi_6{}^2}$$

$$\sqrt{x} = \sqrt{\left(x_2 + R_{s_x}\right)^2 + \left(x_3 + R_{s_y}\right)^2 + \left(x_5 + R_{s_z}\right)^2}$$

$$R_s(t) = \sqrt{R_{s_x}{}^2(t) + R_{s_y}{}^2(t) + R_{s_z}{}^2(t)} \quad \text{(as given in appendix B)}$$

and

$$\rho = \frac{\sqrt{\psi}}{x_7} - \frac{(\psi_7 - \psi_o)}{c}$$

Because $\sqrt{x}$ is bounded away from zero, $\bar{f} = \text{col}(f_1, \ldots f_7)$ and $\bar{g} = \text{col}(g_1, \ldots, g_7)$ are continuous in $\bar{x}$, $\bar{\psi}$, and $\text{sgn}\,\rho$ and piecewise continuous in t with points of discontinuity occurring at the switching points of the switching function ρ.

In the notation of the algorithm, the boundary-value problem presented in appendix C becomes

$$\begin{array}{ll} \alpha_1 = \psi_1(t_o) & e_1 = x_1(t_f) \\ \alpha_2 = \psi_2(t_o) & e_2 = x_2(t_f) \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \alpha_6 = \psi_6(t_o) & e_6 = x_6(t_f) \end{array}$$

with $\psi_7(t_o) - \psi_o$ normalized and $t_f \epsilon \left[t; \rho(t) = 0, \dot{\rho}(t) \leqq 0\right]$. Basically, a problem such as case 3 exists and is solved by the method of case 3(a).

The equations corresponding to equations (15) and (17) for this problem are $\frac{\partial x_i}{\partial \alpha_j}(t) = \frac{\partial x_i}{\partial \alpha_j}(t_o) + \int_{t_o}^{t} \frac{\partial x_{i+1}}{\partial \alpha_j}\,ds$, $\frac{\partial x_i}{\partial \alpha_j}(t_o) = 0$ $(j = 1, 2, \ldots 6;\ i = 1, 3, 5)$. The jumps of f_i $(i = 2, 4, 6, 7)$ and g_7 as t passes through a switching point of ρ are $-\text{sgn}\,\rho(t^{*-}) \frac{\gamma \psi_i(t^*)}{x_7(t^*)\sqrt{\psi(t^*)}}$ $(i = 2, 4, 6)$, $-\frac{\gamma}{c}\,\text{sgn}\,\rho(t^{*-})$ $(i = 7)$, and $-\frac{\text{sgn}\,\rho(t^{*-})\gamma\sqrt{\psi(t^*)}}{x_7^2(t^*)}$ for g_7. Therefore, the remaining equations corresponding to equations (15) and (17) for $j = 1, 2, \ldots 6$ are

$$\frac{\partial x_2(t)}{\partial \alpha_j} = \frac{\partial x_2(t_o)}{\partial \alpha_j} - \gamma \sum_{k=1}^{v} \frac{\text{sgn}\,\rho(t_k^{*-})\psi_2(t_k^*)}{x_7(t_k^*)\sqrt{\psi(t_k^*)}} \frac{\frac{d^{p-1}}{dt^{p-1}} \frac{\partial \rho(t_k^*)}{\partial \alpha_j}}{\frac{d\rho^p}{dt^p}(t_k^*)} H(t - t_k^*)$$

$$+ \int_{t_o}^{t} \left\{ \frac{\gamma}{2}(1 + \text{sgn}\,\rho) \left[\left(\frac{1}{x_7\sqrt{\psi}} - \frac{\psi_2^2}{x_7\sqrt{\psi}^3} \right) \frac{\partial \psi_2}{\partial \alpha_j} - \frac{\psi_2\psi_4 \frac{\partial \psi_4}{\partial \alpha_j}}{x_7\sqrt{\psi}^3} - \frac{\psi_2\psi_6 \frac{\partial \psi_6}{\partial \alpha_j}}{x_7\sqrt{\psi}^3} \right] \right.$$

(Equation continued on next page)

$$
+\left[\frac{3\Omega^2 R_s{}^3 (x_1 + R_{s_x})^2}{\sqrt{x}^5} - \frac{\Omega^2 R_s{}^3}{\sqrt{x}^3} + \omega^2\right]\frac{\partial x_1}{\partial \alpha_j} + \left[3\Omega^2 R_s{}^3 \frac{(x_1 + R_{s_x})(x_3 + R_{s_y})}{\sqrt{x}^5}\right]\frac{\partial x_3}{\partial \alpha_j}
$$

$$
\left. + 2\omega \frac{\partial x_4}{\partial \alpha_j} + \left[3\Omega^2 R_s{}^3 \frac{(x_1 + R_{s_x})(x_5 + R_{s_z})}{\sqrt{x}^5}\right]\frac{\partial x_5}{\partial \alpha_j} - \frac{\frac{\gamma}{2}(1 + \operatorname{sgn}\rho)\psi_2}{x_7{}^2\sqrt{\psi}}\frac{\partial x_7}{\partial \alpha_j}\right\} ds
$$

$$
\left(\frac{\partial x_2}{\partial \alpha_j}(t_o) = 0\right)
$$

$$
\frac{\partial x_4(t)}{\partial \alpha_j} = \frac{\partial x_4(t_o)}{\partial \alpha_j} - \gamma \sum_{k=1}^{v} \frac{\operatorname{sgn}\rho(t_k^{*-})\psi_4(t_k^*)}{x_7(t_k^*)\sqrt{\psi(t_k^*)}} \frac{\frac{d^{p-1}}{dt^{p-1}}\frac{\partial \rho(t_k^*)}{\partial \alpha_j}}{\frac{d^p}{dt^p}\rho(t_k^*)} H(t - t_k^*)
$$

$$
+ \int_{t_o}^{t} \left\{\frac{\gamma}{2}(1 + \operatorname{sgn}\rho)\left[\frac{-\psi_2\psi_4}{x_7\sqrt{x}^3}\frac{\partial \psi_2}{\partial \alpha_j} + \left(\frac{1}{x_7\sqrt{\psi}} - \frac{\psi_4{}^2}{x_7\sqrt{\psi}^3}\right)\frac{\partial \psi_4}{\partial \alpha_j} - \frac{\psi_4\psi_6}{x_7\sqrt{\psi}^3}\frac{\partial \psi_6}{\partial \alpha_j}\right]\right.
$$

$$
+ \left[3\Omega^2 R_s{}^3 \frac{(x_1 + R_{s_x})(x_3 + R_{s_y})}{\sqrt{x}^5}\right]\frac{\partial x_1}{\partial \alpha_j} - 2\omega \frac{\partial x_2}{\partial \alpha_j} + \left[3\Omega^2 R_s{}^3 \frac{(x_3 + R_{s_y})^2}{\sqrt{x}^5}\right.
$$

$$
\left. - \frac{\Omega^2 R_s{}^3}{\sqrt{x}^3} + \omega^2\right]\frac{\partial x_3}{\partial \alpha_j} + \left[3\Omega^2 R_s{}^3 \frac{(x_3 + R_{s_y})(x_5 + R_{s_z})}{\sqrt{x}^5}\right]\frac{\partial x_5}{\partial \alpha_j}
$$

$$
\left. - \frac{\frac{\gamma}{2}(1 + \operatorname{sgn}\rho)\psi_4}{x_7{}^2\sqrt{\psi}}\frac{\partial x_7}{\partial \alpha_j}\right\} ds \qquad \left(\frac{\partial x_4}{\partial \alpha_j}(t_o) = 0\right)
$$

$$
\frac{\partial x_6(t)}{\partial \alpha_j} = \frac{\partial x_6(t_o)}{\partial \alpha_j} - \gamma \sum_{k=1}^{v} \frac{\operatorname{sgn}\rho(t_k^{*-})\psi_6(t_k^*)}{x_7(t_k^*)\sqrt{\psi(t_k^*)}} \frac{\frac{d^{p-1}}{dt^{p-1}}\frac{\partial \rho(t_k^*)}{\partial \alpha_j}}{\frac{d^p}{dt^p}\rho(t_k^*)} H(t - t_k^*) + \int_{t_o}^{t} \left\{\frac{\gamma}{2}(1 + \operatorname{sgn}\rho)\right.
$$

(Equation continued on next page)

$$\times\left[\frac{-\psi_2\psi_6}{x_7\sqrt{\psi}^3}\frac{\partial\psi_2}{\partial\alpha_j} - \frac{\psi_4\psi_6}{x_7\sqrt{\psi}^3}\frac{\partial\psi_4}{\partial\alpha_j} + \left(\frac{1}{x_7\sqrt{\psi}} - \frac{\psi_6^2}{x_7\sqrt{\psi}^3}\right)\frac{\partial\psi_6}{\partial\alpha_j}\right] + \left[3\Omega^2 R_s{}^3\frac{\left(x_1 + R_{s_x}\right)\left(x_5 + R_{s_z}\right)}{\sqrt{x}^5}\right]\frac{\partial x_1}{\partial\alpha_j}$$

$$+\left[3\Omega^2 R_s{}^3\frac{\left(x_3 + R_{s_y}\right)\left(x_5 + R_{s_z}\right)}{\sqrt{x}^5}\right]\frac{\partial x_3}{\partial\alpha_j} + \left[3\Omega^2 R_s{}^3\frac{\left(x_5 + R_{s_z}\right)^2}{\sqrt{x}^5} - \frac{\Omega^2 R_s{}^3}{\sqrt{x}^3}\right]\frac{\partial x_5}{\partial\alpha_j}$$

$$\left.-\frac{\frac{\gamma}{2}(1 + \text{sgn}\,\rho)\psi_6}{x_7{}^2\sqrt{\psi}}\frac{\partial x_7}{\partial\alpha_j}\right\}ds \qquad \left(\frac{\partial x_6}{\partial\alpha_j}(t_o) = 0\right)$$

$$\frac{\partial x_7(t)}{\partial\alpha_j} = \frac{\partial x_7}{\partial\alpha_j}(t_o) + \frac{\gamma}{c}\sum_{k=1}^{v}\text{sgn}\,\rho\left(t_k^{*-}\right)\frac{\frac{d^{p-1}}{dt^{p-1}}\frac{\partial\rho\left(t_k^*\right)}{\partial\alpha_j}}{\frac{d\rho^p}{dt^p}\left(t_k^*\right)}H(t - t^*) \qquad \left(\frac{\partial x_7}{\partial\alpha_j}(t_o) = 0\right)$$

$$\frac{\partial\psi_1(t)}{\partial\alpha_j} = \frac{\partial\psi_1(t_o)}{\partial\alpha_j} + \int_{t_o}^{t}\left\{\left[\frac{\Omega^2 R_s{}^3}{\sqrt{x}^3} - \frac{3\Omega^2 R_s{}^3\left(x_1 + R_{s_x}\right)^2}{\sqrt{x}^5} - \omega^2\right]\frac{\partial\psi_2}{\partial\alpha_j}\right.$$

$$-\left[3\Omega^2 R_s{}^3\frac{\left(x_1 + R_{s_x}\right)\left(x_3 + R_{s_y}\right)}{\sqrt{x}^5}\right]\frac{\partial\psi_4}{\partial\alpha_j} - \left[3\Omega^2 R_s{}^3\frac{\left(x_5 + R_{s_z}\right)\left(x_1 + R_{s_x}\right)}{\sqrt{x}^5}\right]\frac{\partial\psi_6}{\partial\alpha_j}$$

$$+\left[-3\Omega^2 R_s{}^3\frac{2\psi_2\left(x_1 + R_{s_x}\right) + d}{\sqrt{x}^5} + 15\Omega^2\frac{R_s{}^3 d\left(x_1 + R_{s_x}\right)^2}{\sqrt{x}^7}\right]\frac{\partial x_1}{\partial\alpha_j}$$

$$+\left[-3\Omega^2 R_s{}^3\frac{\psi_2\left(x_3 + R_{s_y}\right) + \psi_4\left(x_1 + R_{s_x}\right)}{\sqrt{x}^5} + \frac{15d\Omega^2 R_s{}^3}{\sqrt{x}^7}\left(x_1 + R_{s_y}\right)\left(x_3 + R_{s_y}\right)\right]\frac{\partial x_3}{\partial\alpha_j}$$

$$\left.+\left[-3\Omega^2 R_s{}^3\frac{\psi_2\left(x_5 + R_{s_z}\right) + \psi_6\left(x_1 + R_{s_x}\right)}{\sqrt{x}^5} + 15d\Omega^2 R_s{}^3\frac{\left(x_1 + R_{s_x}\right)\left(x_5 + R_{s_z}\right)}{\sqrt{x}^7}\right]\frac{\partial x_5}{\partial\alpha_j}\right\}ds$$

$$\left(\frac{\partial\psi_1}{\partial\alpha_j}(t_o) = \begin{cases}1 & (j = 1)\\ 0 & (\text{Otherwise})\end{cases}\right)$$

$$\frac{\partial\psi_2(t)}{\partial\alpha_j} = \frac{\partial\psi_2(t_o)}{\partial\alpha_j} + \int_{t_o}^{t}\left(2\omega\ \frac{\partial\psi_4}{\partial\alpha_j} - \frac{\partial\psi_1}{\partial\alpha_j}\right)ds \qquad \left(\frac{\partial\psi_2(t_o)}{\partial\alpha_j} = \begin{cases}1 & (j = 2)\\ 0 & (\text{Otherwise})\end{cases}\right)$$

$$\frac{\partial\psi_3(t)}{\partial\alpha_j} = \frac{\partial\psi_3(t_o)}{\partial\alpha_j} + \int_{t_o}^{t}\left(\left[-3\Omega^2 R_s{}^3\ \frac{\left(x_1 + R_{s_x}\right)\left(x_3 + R_{s_y}\right)}{\sqrt{x}^{\,5}}\right]\frac{\partial\psi_2}{\partial\alpha_j}\right.$$

$$+ \left[\frac{\Omega^2 R_s{}^3}{\sqrt{x}^{\,3}} - 3\Omega^2 R_s{}^3\ \frac{\left(x_3 + R_{s_y}\right)^2}{\sqrt{x}^{\,5}} - \omega^2\right]\frac{\partial\psi_4}{\partial\alpha_j} + \left[-3\Omega^2 R_s{}^3\ \frac{\left(x_3 + R_{s_y}\right)\left(x_5 + R_{s_z}\right)}{\sqrt{x}^{\,5}}\right]\frac{\partial\psi_6}{\partial\alpha_j}$$

$$+ \left[-3\Omega^2 R_s{}^3\ \frac{\psi_4\left(x_1 + R_{s_x}\right) + \psi_2\left(x_3 + R_{s_y}\right)}{\sqrt{x}^{\,5}} + 15\Omega^2 R_s{}^3 d\ \frac{\left(x_1 + R_{s_x}\right)\left(x_3 + R_{s_y}\right)}{\sqrt{x}^{\,7}}\right]\frac{\partial x_1}{\partial\alpha_j}$$

$$+ \left[-3\Omega^2 R_s{}^3\ \frac{2\psi_4\left(x_3 + R_{s_y}\right) + d}{\sqrt{x}^{\,5}} + \frac{15\Omega^2 R_s{}^3 d\left(x_3 + R_{s_y}\right)^2}{\sqrt{x}^{\,7}}\right]\frac{\partial x_3}{\partial\alpha_j}$$

$$\left. + \left\{-\frac{3\Omega^2 R_s{}^3}{\sqrt{x}^{\,5}}\left[\psi_4\left(x_5 + R_{s_z}\right) + \psi_6\left(x_3 + R_{s_y}\right)\right] + 15\Omega^2 R_s{}^3 d\ \frac{\left(x_3 + R_{s_y}\right)\left(x_5 + R_{s_z}\right)}{\sqrt{x}^{\,7}}\right\}\frac{\partial x_5}{\partial\alpha_j}\right)ds$$

$$\left(\frac{\partial\psi_3}{\partial\alpha_j} = \begin{cases}1 & (j = 3)\\ 0 & (\text{Otherwise})\end{cases}\right)$$

$$\frac{\partial\psi_4(t)}{\partial\alpha_j} = \frac{\partial\psi_4(t_o)}{\partial\alpha_j} - \int_{t_o}^{t}\left(2\omega\ \frac{\partial\psi_2}{\partial\alpha_j} + \frac{\partial\psi_3}{\partial\alpha_j}\right)ds \qquad \left(\frac{\partial\psi_4}{\partial\alpha_j}(t_o) = \begin{cases}1 & (j = 4)\\ 0 & (\text{Otherwise})\end{cases}\right)$$

$$\frac{\partial\psi_5(t)}{\partial\alpha_j} = \frac{\partial\psi_5(t_o)}{\partial\alpha_j} + \int_{t_o}^{t}\left(\left[-\frac{3\Omega^2 R_s{}^3}{\sqrt{x}^{\,5}}\left(x_1 + R_{s_x}\right)\left(x_5 + R_{s_z}\right)\right]\frac{\partial\psi_2}{\partial\alpha_j}\right.$$

$$+ \left[-3\Omega^2 R_s{}^3\ \frac{\left(x_3 + R_{s_y}\right)\left(x_5 + R_{s_z}\right)}{\sqrt{x}^{\,5}}\right]\frac{\partial\psi_4}{\partial\alpha_j} + \left[\frac{\Omega^2 R_s{}^3}{\sqrt{x}^{\,3}} - 3\Omega^2 R_s{}^3\ \frac{\left(x_5 + R_{s_z}\right)^2}{\sqrt{x}^{\,5}}\right]\frac{\partial\psi_6}{\partial\alpha_j}$$

(Equation continued on next page)

$$+\left\{-3\Omega^2 R_s{}^3\left[\frac{\psi_6\left(x_1 + R_{s_x}\right) + \psi_2\left(x_5 + R_{s_z}\right)}{\sqrt{x}^{\,5}}\right] + 15\Omega^2 R_s{}^3 d\,\frac{\left(x_1 + R_{s_x}\right)\left(x_5 + R_{s_z}\right)}{\sqrt{x}^{\,7}}\right\}\frac{\partial x_1}{\partial \alpha_j}$$

$$+\left[-3\Omega^2 R_s{}^3\,\frac{\psi_6\left(x_3 + R_{s_y}\right) + \psi_4\left(x_5 + R_{s_z}\right)}{\sqrt{x}^{\,5}} + 15\Omega^2 R_s{}^3 d\,\frac{\left(x_3 + R_{s_y}\right)\left(x_5 + R_{s_z}\right)}{\sqrt{x}^{\,7}}\right]\frac{\partial x_3}{\partial \alpha_j}$$

$$+\left[-3\Omega^2 R_s{}^3\,\frac{2\psi_6\left(x_5 + R_{s_z}\right) + d}{\sqrt{x}^{\,5}} + 15\Omega^2 R_s{}^3 d\,\frac{\left(x_5 + R_{s_z}\right)^2}{\sqrt{x}^{\,7}}\right]\frac{\partial x_5}{\partial \alpha_j}\Bigg) ds$$

$$\left(\frac{\partial \psi_5(t_o)}{\partial \alpha_j} = \begin{cases} 1 & (j = 5) \\ 0 & (\text{Otherwise}) \end{cases}\right)$$

$$\frac{\partial \psi_6(t)}{\partial \alpha_j} = \frac{\partial \psi_6}{\partial \alpha_j}(t_o) - \int_{t_o}^{t} \frac{\partial \psi_5}{\partial \alpha_j}\, ds \qquad \left(\frac{\partial \psi_6}{\partial \alpha_j} = \begin{cases} 1 & (j = 6) \\ 0 & (\text{Otherwise}) \end{cases}\right)$$

$$\frac{\partial \psi_7(t)}{\partial \alpha_j} = \frac{\partial \psi_7(t_o)}{\partial \alpha_j} - \gamma \sum_{k=1}^{v} \frac{\operatorname{sgn} \rho\left(t_k^{*-}\right)\sqrt{\psi\left(t_k^*\right)}}{x_7{}^2\left(t_k^*\right)}\;\frac{\dfrac{d^{p-1}}{dt^{p-1}}\dfrac{\partial \rho\left(t_k^*\right)}{\partial \alpha_j}}{\dfrac{d^p}{dt^p}\rho\left(t_k^*\right)}\; H\left(t - t_k^*\right)$$

$$+\int_{t_o}^{t} \frac{\gamma}{2}(1 + \operatorname{sgn} \rho)\left(\frac{\psi_2 \dfrac{\partial \psi_2}{\partial \alpha_j} + \psi_4 \dfrac{\partial \psi_4}{\partial \alpha_j} + \psi_6 \dfrac{\partial \psi_6}{\partial \alpha_j}}{x_7{}^2\sqrt{\psi}} - \frac{2\sqrt{\psi}}{x_7{}^3}\frac{\partial x_7}{\partial \alpha_j}\right) ds \qquad \left(\frac{\partial \psi_7}{\partial \alpha_j}(t_o) = 0\right)$$

The functions $\partial f_i/\partial \alpha_j$ and $\partial g_i/\partial \alpha_j$ are the integrands in the equations for $\partial x_i/\partial \alpha_j$ and $\partial \psi_i/\partial \alpha_j$ $(i = 1, 2, \ldots 7;\ j = 1, 2, \ldots 6)$. These are bounded and continuous on $J(t) = [t_o, t] - S(t^*)$ where $S(t^*)$ is the set of switching points of $\rho(t)$ within $[t_o, t_f]$.

Also

$$\frac{d\rho}{dt} = \frac{\frac{d}{dt}\sqrt{\psi_2^{\,2} + \psi_4^{\,2} + \psi_6^{\,2}}}{x_7} - \frac{\sqrt{\psi_2^{\,2} + \psi_4^{\,2} + \psi_6^{\,2}}}{x_7^{\,2}}\frac{dx_7}{dt} - \frac{\frac{d\psi_7}{dt}}{c} = \frac{\psi_2\dot{\psi}_2 + \psi_4\dot{\psi}_4 + \psi_6\dot{\psi}_6}{x_7\sqrt{\psi}} - \frac{\sqrt{\psi}u_4}{cx_7^{\,2}}$$

$$+ \frac{\sqrt{\psi}u_4}{cx_7^{\,2}} = - \frac{\psi_1\psi_2 + \psi_3\psi_4 + \psi_5\psi_6}{x_7\sqrt{\psi}} = - \frac{\psi_1 u_1 + \psi_3 u_2 + \psi_5 u_3}{x_7}$$

and, thus, $\frac{d\rho}{dt}(t)$ is a continuous function of t. Also

$$\frac{\partial\rho}{\partial\alpha_j} = \frac{\psi_2\frac{\partial\psi_2}{\partial\alpha_j} + \psi_4\frac{\partial\psi_4}{\partial\alpha_j} + \psi_6\frac{\partial\psi_6}{\partial\alpha_j}}{x_7\sqrt{\psi}} - \frac{\sqrt{\psi}}{x_7^{\,2}}\frac{\partial x_7}{\partial\alpha_j} - \frac{\frac{\partial\psi_7}{\partial\alpha_j}}{c}$$

Substituting for $\frac{\partial\psi_7}{\partial\alpha_j}(t)$ and $\frac{\partial x_7}{\partial\alpha_j}(t)$ yields

$$\frac{\partial\rho}{\partial\alpha_j}(t) = C(t) + \sum_{k=1}^{v} \operatorname{sgn}\rho\left(t_k^{*-}\right)\frac{\gamma}{c}\frac{\frac{d^{p-1}}{dt^{p-1}}\frac{\partial\rho\left(t_k^*\right)}{\partial\alpha_j}}{\frac{d^p}{dt^p}\rho(t^*)}\left[-\frac{\sqrt{\psi}(t)}{x_7^{\,2}(t)} + \frac{\sqrt{\psi}\left(t_k^*\right)}{x_7^{\,2}\left(t_k^*\right)}\right]H\left(t - t_k^*\right)$$

where

$$C(t) = \frac{\psi_2\frac{\partial\psi_2}{\partial\alpha_j} + \psi_4\frac{\partial\psi_4}{\partial\alpha_j} + \psi_6\frac{\partial\psi_6}{\partial\alpha_j}}{\sqrt{\psi}(t)x_7(t)} - \frac{\sqrt{\psi}(t)}{x_7^{\,2}(t)}\left.\frac{\partial x_1}{\partial\alpha_j}(t)\right|_C - \left.\frac{\partial\psi_7}{\partial\alpha_j}\right|_C$$

The symbols $\left.\frac{\partial x_7}{\partial\alpha_j}\right|_C$ and $\left.\frac{\partial\psi_7}{\partial\alpha_j}\right|_C$ refer to the continuous parts of $\frac{\partial x_7}{\partial\alpha_j}(t)$ and $\frac{\partial\psi_7}{\partial\alpha_j}(t)$, respectively, and result when the terms multiplying $H\left(t - t_k^*\right)$ are dropped in the equations for $\frac{\partial x_7}{\partial\alpha_j}$ and $\frac{\partial\psi_7}{\partial\alpha_j}$.

Even though $H\left(t - t_k^*\right)$ is not defined at t^*, $H\left(t - t_k^*\right)$ is assumed to be bounded by one for all $t_o \leqq t \leqq t_f$; consequently, the definition $\frac{\partial\rho}{\partial\alpha_j}\left(t_k^*\right) = \frac{\partial\rho}{\partial\alpha_j}\left(t_k^{*-}\right)$ renders $\frac{\partial\rho}{\partial\alpha_j}(t)$

continuous at t_k^*. Time derivatives of $\frac{\partial \rho(t)}{\partial \alpha_j}$ are not continuous at t_k^* because $\frac{d}{dt} H(t - t^*)$ is not bounded. In order to apply the algorithm, only the cases where the switching points of $\rho(t)$ are simple zeros can be considered. For all such cases considered, the number of zeros of $\rho(t)$ were finite.

Finally, because $\bar{\alpha} = (\alpha_1, \ldots \alpha_6)'$ and $\bar{e}(\bar{\alpha},t_f) = (e_1, \ldots e_6)'$, the matrix $\frac{\partial \bar{e}}{\partial \bar{\alpha}}(\bar{\alpha},t_f)$ is the (6×6) array $(\partial x_i/\partial \alpha_j)$ $(i = 1, 2, \ldots 6;\ j = 1, 2, \ldots 6)$. The measure of terminal error is $E(\bar{\alpha},t_f) = \sum_{i=1}^{6} \frac{b_i x_i(t_f)}{2}$.

Results

The algorithm and system equations were programed for the IBM 7094 electronic data processing system by using the Fortran IV language. Copies of the program are available on request from the Trajectory Applications Section, Langley Research Center, for the problem "Fuel Optimal Rendezvous" (program no. E1257). Integration was performed with fixed-step sizes of either 2 or 4 seconds by using a method with a fourth-order Adams-Bashforth predictor formula and a fourth-order Adams-Moulton corrector formula.

The program was such that fixed-final-time and free-final-time solutions could be obtained. The program had the option of iteration at a fixed time or at a time when, after a specified number of coast periods have taken place, $\rho = 0$ and $\dot{\rho} \leqq 0$. The approach taken in constructing free-time solutions was to begin with a nominal and to compute successively fixed-time solutions for increasing values of the final time until, at such a time, a zero of $\rho(t)$ was observed, which satisfied a prescribed number of coasts with $\dot{\rho} \leqq 0$. This solution was then used as a nominal with $\rho(t) = 0$ and $\dot{\rho} \leqq 0$ as a stopping condition.

The satellite orbital plane was placed in the xy-plane of the rotating system. (See fig. B-1 of appendix B.) Examples were computed both with the vehicle launched from absolute rest from the surface of the moon in the satellite plane and from absolute rest from out of plane. Table I shows the values of the fixed parameters for classes of both examples.

TABLE I.- FIXED PARAMETERS

Initial time, t_o	0 sec
Upper bound on thrust magnitude, γ	3504 lbm (1589.4 kg)
Initial mass, m_o	285.5 slugs (4166.3 kg)
Effective exhaust velocity, c	9853.2 ft/sec (279.8 m/sec)
Radius of moon, $R_V(t_o)$	5.707×10^6 ft (1739.4 km)
Radius of satellite orbit,[1] R_S	6.1934×10^6 ft (1887.7 km)
Gravitational parameter of moon, μ	1.727×10^{14} ft^3/sec^2 $(48.9 \times 10^{11}$ m^3/sec$^2)$
Angular velocity of moon about axis of rotation, ω	2.66×10^{-6} rad/sec

[1]Satellite is in an 80 n. mi. circular orbit.

It was found that a workable set of b_i (i = 1, 2, . . . 6) and $|\lambda|$ for convergence was $b_1 = b_3 = b_5 = 1$, $b_2 = b_4 = b_6 = 10$, and $|\lambda| = 10$. Except as noted, these values were used throughout. An increase in the value of $|\lambda|$ yielded slower convergence; whereas, a decrease was apt to produce divergence. Greater influence could be applied to the correction of the error e_i by increasing a particular b_i; that is, this error would be corrected more quickly than before, probably at the expense of the other errors. A similar statement could be made for the lack of influence on correcting e_i observed by decreasing b_i.

In-plane results are presented in table II. When the vehicle was launched such that the satellite lead angle $\varphi_o - \varphi_v^o$ was 13.7° (φ_o = 89°), the set of values

$$\psi_o = 0$$

$$\psi_1(t_o) = 2.0$$

$$\psi_2(t_o) = 3000.0$$

$$\psi_3(t_o) = 10.0$$

$$\psi_4(t_o) = 3000.0$$

$$\psi_5(t_o) = 0$$

$$\psi_6(t_o) = 0$$

$$\psi_7(t_o) = -3.3 \times 10^6$$

was found to yield a trajectory which continuously gained altitude with $\rho(t) > 0$ throughout; that is, no switching points were encountered. Then, ψ_o was reset such that $\rho(t)$ went through a zero; that is, the vehicle began to coast at about 250 seconds. These values then produced a nominal set of values for $\alpha_1 - \alpha_6$ to which successive

TABLE II.- FUEL OPTIMAL IN-PLANE RESULTS

$\left[\varphi_o - \varphi_v{}^o = 13.7^o\right]$

Final time, t_f, sec	First coast time, sec	Final burn time, sec	Percent of initial mass at t_f	Unknown parameters					
				$\psi_o \times 10^{-6}$	$\psi_1(t_o)$	$\psi_2(t_o)$	$\psi_3(t_o)$	$\psi_4(t_o)$	$\psi_7(t_o) \times 10^{-5}$
620.0	313.5	539.1	51.07	-0.19525	2.43870	3708.9	10.6080	3195.4	1.12350
720.0	321.6	657.9	52.12	-.18058	1.54280	3300.4	7.5372	2332.2	-.97578
850.0	328.7	800.0	52.83	-.17123	.86229	2991.7	5.7089	1801.4	-.88233
1000.0	334.6	959.2	53.24	-.16536	.37087	2766.5	4.6379	1477.4	-.82356
1150.0	339.0	1115.1	53.43	-.16189	.05115	2617.9	4.0602	1291.6	-.78890
1300.0	342.5	1269.1	53.49	-.15974	-.16030	2518.1	3.7377	1177.5	-.76740
1350.0	343.5	1320.8	53.50	-.15946	-.18535	2506.1	3.6927	1156.3	-.76464
[a]1390.6	344.3	1361.6	53.51	-.15912	-.21928	2489.8	3.6482	1137.3	-.76122

[a]Free-final-time solution, $|\rho(1390.6)| = 0.432 \times 10^{-6}$.

application of the correction equation (10) at a fixed-final time of 620.0 seconds yielded the first entry in table II. Other solutions were computed by using the nearest obtained solution in final time as a nominal. At the 1350.0-second case, a zero of $\rho(t)$ which had the property $\dot{\rho}(t) < 0$ after one coasting period was predicted. With $\rho(t_f) = 0$ and $\dot{\rho}(t_f) < 0$ as stopping conditions, the 1350.0-second case as a nominal, and $|\lambda| = 10^4$, the 1390.6-second case resulted.

For each of the solutions below the 1150.0-second entry, the trajectories remained below the altitude of the satellite orbit. Near the 1150.0-second final time, the trajectories, during the coast phase, began to reach an altitude which was higher than that of the target orbit. Such a property is referred to as "overshoot." Trajectories with and without overshoot are contrasted in figures 1 to 3. The trajectory without overshoot is the 850.0-second entry, and the one with overshoot is the free-final-time 1390.6-second solution. Arrows along the solid-line burning portions of the trajectories indicate the direction of the thrust vector. Dotted-line portions indicate coasting periods. The xy-axis system is represented as inertial because the total time of flight is such that the angular displacement of the moon about its own axis is less than 0.3^o.

Table III shows a sample iteration for the fixed-final-time 1300.0-second solution when the 1000.0-second solution is used as a nominal. Results for the vehicle launched 5^o out of the target orbital plane are presented in table IV. Overshoot begins near the 1200.0-second entry. In-plane results are used for beginning nominals.

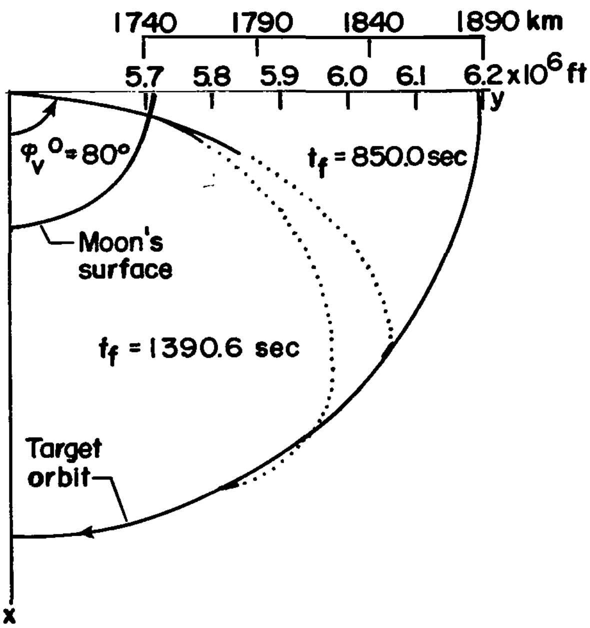

Figure 1.- In-plane trajectories for $\varphi_0 = 93.7^0$ to final times of 850.0 and 1390.6 seconds.

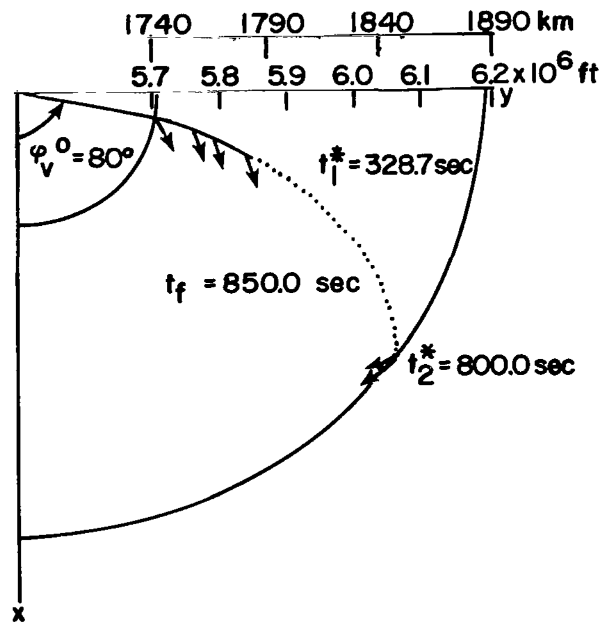

Figure 2.- In-plane trajectory for $\varphi_0 = 93.7^0$ to final time of 850.0 seconds.

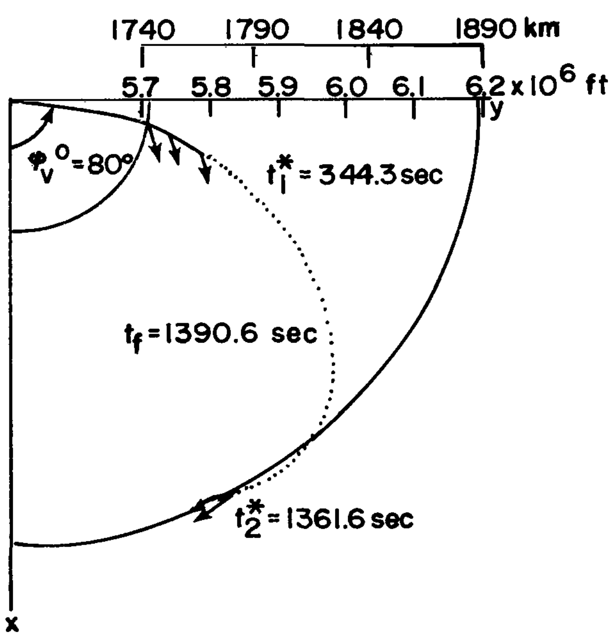

Figure 3.- In-plane trajectory for $\varphi_0 = 93.7^0$ to final time of 1390.6 seconds.

Thrust portions of trajectory

Coast portion of trajectory

Direction of thrust vector

TABLE III.- FUEL-OPTIMAL ITERATION SEQUENCE

$\left[\varphi_o - \varphi_v{}^o = 13.7^o;\ \ \theta_v{}^o = 0^o;\ \ t_f = 1300.0 \text{ sec}\right]$

Iteration	First coast time, sec	Final burn time, sec	Unknown parameters				Error criterion $E(\bar{\alpha},t_f)$
			$\psi_1(t_o)$	$\psi_2(t_o)$	$\psi_3(t_o)$	$\psi_4(t_o)$	
[a]0	334.6	959.2	0.370870	2766.5	4.6379	1477.5	0.101×10^{13}
1	360.8	1296.1	.401920	2710.0	3.9214	1091.3	$.129 \times 10^{12}$
2	343.9	1268.7	.291960	2734.2	4.0687	1256.0	$.776 \times 10^{9}$
3	342.3	1259.2	.145090	2663.0	3.9937	1244.8	$.306 \times 10^{8}$
4	342.6	1262.5	.036295	2611.7	3.9098	1221.7	$.228 \times 10^{8}$
5	342.5	1264.7	-.035214	2577.6	3.8512	1206.4	$.726 \times 10^{7}$
6	342.5	1266.5	-.081362	2555.7	3.8110	1196.1	$.234 \times 10^{7}$
7	342.5	1267.7	-.110730	2541.7	3.7844	1189.3	$.596 \times 10^{6}$
8	342.5	1268.4	-.129310	2532.8	3.7672	1184.9	$.137 \times 10^{6}$
9	342.5	1268.8	-.141050	2527.2	3.7561	1182.1	$.290 \times 10^{5}$
10	342.5	1269.0	-.148460	2523.7	3.7491	1180.4	$.563 \times 10^{4}$
11	342.5	1269.1	-.153140	2321.5	3.7446	1179.2	$.105 \times 10^{4}$
12	342.5	1269.1	-.156100	2520.0	3.7418	1178.5	$.204 \times 10^{3}$
13	342.5	1269.1	-.157940	2519.2	3.7400	1178.0	$.338 \times 10^{2}$
14	342.5	1269.1	-.159100	2518.6	3.7389	1177.8	$.120 \times 10^{2}$
15	342.5	1269.1	-.159840	2518.3	3.7382	1177.6	$.167 \times 10$
[b]16	342.5	1269.1	-.160300	2518.1	3.7377	1177.5	.600

[a]Nominal $t_f = 1000.0$ sec; $x_1(t_f) = 0.7616 \times 10^6$, $x_2(t_f) = 0.52222 \times 10^4$, $x_3(t_f) = -0.12316 \times 10^7$, and $x_4(t_f) = -0.10558 \times 10^5$.

[b]$x_1(t_f) = -0.18241$, $x_2(t_f) = -0.23426$, $x_3(t_f) = -0.36939$, and $x_4(t_f) = -0.21920$.

TABLE IV.- FUEL-OPTIMAL OUT-OF-PLANE RESULTS

$\left[\varphi_o = 93.7^o;\ \ \varphi_v{}^o = 80^o;\ \ \theta_v = 5^o\right]$

Final time, t_f, sec	First coast time, sec	Final burn time, sec	Percent of initial mass at t_f	Unknown parameters							
				$\psi_o \times 10^{-6}$	$\psi_1(t_o)$	$\psi_2(t_o)$	$\psi_3(t_o)$	$\psi_4(t_o)$	$\psi_5(t_o)$	$\psi_6(t_o)$	$\psi_7(t_o) \times 10^{-5}$
620.0	324.6	528.4	48.16	-0.23492	4.320300	4718.2	12.3090	3937.9	-6.6467	-2179.00	-1.51920
720.0	327.5	645.1	49.88	-.20325	2.805700	3928.8	8.1004	2657.5	-4.1913	-1372.40	-1.20250
850.0	331.7	789.5	51.13	-.18462	1.649800	3368.9	5.7369	1921.5	-2.8230	-892.25	-1.01620
1000.0	336.2	945.0	51.88	-.17456	.939600	3032.4	4.5158	1525.5	-2.0924	-630.00	-.91560
1100.0	338.8	1054.8	52.17	-.17063	.636920	2889.6	4.0662	1371.4	-1.8051	-522.38	-.87632
1200.0	341.0	1158.6	52.36	-.16790	.415080	2784.7	3.7718	1246.0	-1.6015	-443.16	-.83442
1300.0	343.1	1261.6	52.47	-.16557	.250510	2706.5	3.5776	1186.9	-1.3925	-382.35	-.82973
1400.0	344.9	1363.9	52.54	-.16460	.128460	2648.0	3.4511	1130.3	-1.3410	-334.20	-.81598
1500.0	346.5	1465.8	52.57	-.16362	.038713	2604.4	3.3715	1088.1	-1.2578	-295.18	-.80692
1600.0	348.0	1567.4	52.58	-.16342	.035952	2601.4	3.3406	1062.6	-1.2457	-273.17	-.80483
[a]1614.3	348.3	1582.0	52.58	-.16336	.032755	2599.6	3.3362	1058.9	-1.2425	-269.84	-.80429

[a]Free-final-time solution, $|\rho(1614.3)| = 0.461 \times 10^{-6}$.

Running time for all cases on the IBM 7094 electronic data processing system was approximately 7 minutes. In programing this example, the primary objective was to decide whether the method could be applied to such a nonlinear two-point boundary-value problem and not necessarily to write a program giving solutions in a minimum of computer time. Time-consuming subroutines were included to test for conditions leading to numerical instability (overflow, underflow, and so forth) and to determine the nature of the zeros of the switching function. These subroutines and the large number of equations involved account for the rather long computer time. No sensitivity or convergence problems were found in any of the cases considered.

CONCLUDING REMARKS

A successive approximation procedure for attacking a class of two-point boundary-value problems which frequently occurs in indirect optimization theory has been presented. Basically, the boundary-value problem was one in which the optimal-control law was piecewise continuous and in which there were a number of system parameters to be determined to meet an equal number of terminal conditions. An iterative logic was developed in which an assumed set of parameters would be improved upon so that, by repetitive use of a correction formula, a monotonic decreasing sequence of values of a positive definite function that measures the terminal errors was produced.

The procedure provided several important advantages. A forward integration and a single matrix inversion must be performed to compute the correction vector. The matrix to be inverted was guaranteed to be nonsingular. The direction of the correction vector was found to lie between the direction given by the gradient and the Newton-Raphson procedures. An additional advantage was that a final-time correction could be made an integral part of the process provided that the final time was an unknown parameter; that is, corrections in the final time would be computed at each iteration as a component of the parameter correction vector. Integral equations were derived for influence matrices that describe the effect of a change in the parameters on the terminal conditions.

The process also had some disadvantages. The fact that complete convergence could be obtained from an arbitrary choice of assumed parameters was not established. The technique proposed is basically a boundary-condition iteration scheme. Such schemes generally have sensitivity and convergence problems. The process does not generally eliminate such difficulties, but none were found in the example considered. Finally, the technique cannot be used in problems in which a singular control might occur unless such an occurrence can be predicted.

In order to demonstrate the usefulness of the procedure, solutions were obtained to the two-point boundary-value problem resulting from an application of the Pontryagin maximum principle to obtain fuel-optimal lunar-rendezvous trajectories for a target in a circular orbit. Fixed- and free-final-time solutions were computed for planar and nonplanar situations. Running times on the IBM 7094 electronic data processing system were on the order of 7 minutes. In programing this example, the primary objective was to decide whether the method could be applied to such a nonlinear two-point boundary-value problem and not necessarily to write a program giving solutions in a minimum of computer time. Time-consuming subroutines were included to test for conditions leading to numerical instability (overflow, underflow, and so forth) and to determine the nature of the zeros of the switching function. These subroutines and the large number of equations involved accounted for the rather long computer time.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., May 8, 1968,
125-19-04-01-23.

APPENDIX A

PROOF OF LEMMAS USED TO ESTABLISH THEOREM 1

The lemmas used to establish theorem 1 are now proved.

Lemma 1

Unique solutions $\delta\bar{\alpha}^{o}(\nu)$ and $\lambda(\nu) < 0$ of the system

$$\left.\begin{aligned} &\left\|\delta\bar{\alpha}^{o}\right\|^{2} = \nu^{2} \\ &\delta\bar{\alpha}^{o} = -\left[\frac{\partial\bar{e}'}{\partial\bar{\alpha}}\left(\bar{\alpha}^{o},t_f\right)B\,\frac{\partial\bar{e}}{\partial\bar{\alpha}}\left(\bar{\alpha}^{o},t_f\right) - \lambda I\right]^{-1}\frac{\partial\bar{e}'}{\partial\bar{\alpha}}\left(\bar{\alpha}^{o},t_f\right)B\bar{e}\left(\bar{\alpha}^{o},t_f\right)\end{aligned}\right\} \tag{A1}$$

exist if ν is sufficiently small.

Proof: Note that $\frac{\partial\bar{e}'}{\partial\bar{\alpha}}\left(\bar{\alpha}^{o},t_f\right)B\,\frac{\partial\bar{e}}{\partial\bar{\alpha}}\left(\bar{\alpha}^{o},t_f\right)$ is a real symmetric matrix and can therefore be diagonalized (ref. 12). There exists an orthogonal matrix $A, A' = A^{-1}$, which operates on $\frac{\partial\bar{e}'}{\partial\bar{\alpha}}\left(\bar{\alpha}^{o},t_f\right)B\,\frac{\partial\bar{e}'}{\partial\bar{\alpha}}\left(\bar{\alpha}^{o},t_f\right)$ to yield

$$A\,\frac{\partial\bar{e}'}{\partial\bar{\alpha}}\left(\bar{\alpha}^{o},t_f\right)B\,\frac{\partial\bar{e}}{\partial\bar{\alpha}}\left(\bar{\alpha}^{o},t_f\right)A' = \mathrm{diag}\left(\lambda_i\right) \qquad (i = 1, 2, \ldots m)$$

where the λ_i are the eigenvalues of $\frac{\partial\bar{e}'}{\partial\bar{\alpha}}\left(\bar{\alpha}^{o},t_f\right)B\,\frac{\partial\bar{e}}{\partial\bar{\alpha}}\left(\bar{\alpha}^{o},t_f\right)$. If $\bar{\eta}$ is an arbitrary m-dimensional column vector, then

$$\bar{\eta}\cdot\frac{\partial\bar{e}'}{\partial\bar{\alpha}}\left(\bar{\alpha}^{o},t_f\right)B\,\frac{\partial\bar{e}}{\partial\bar{\alpha}}\left(\bar{\alpha}^{o},t_f\right)\bar{\eta} = \sqrt{B}\,\frac{\partial\bar{e}}{\partial\bar{\alpha}^{o}}\left(\bar{\alpha}^{o},t_f\right)\bar{\eta}\cdot\sqrt{B}\,\frac{\partial\bar{e}}{\partial\bar{\alpha}}\left(\bar{\alpha}^{o},t_f\right)\bar{\eta} \geqq 0$$

where $\sqrt{B} = \mathrm{diag}\sqrt{b_i}$ because $B = \mathrm{diag}(b_i)$ $(i = 1, 2, \ldots m)$. Thus, $\frac{\partial\bar{e}'}{\partial\bar{\alpha}}\left(\bar{\alpha}^{o},t_f\right)B\,\frac{\partial\bar{e}}{\partial\bar{\alpha}}\left(\bar{\alpha}^{o},t_f\right)$ is nonnegative definite and, therefore, has nonnegative eigenvalues (ref. 13); that is, $\lambda_i \geqq 0$ for all $i = 1, 2, \ldots m$. Therefore, the inverse of $\frac{\partial\bar{e}'}{\partial\bar{\alpha}}\left(\bar{\alpha}^{o},t_f\right)B\,\frac{\partial\bar{e}}{\partial\bar{\alpha}}\left(\bar{\alpha}^{o},t_f\right) - \lambda I$ exists because $\lambda < 0$. When

$$\bar{c} = -A\,\frac{\partial\bar{e}'}{\partial\bar{\alpha}}\left(\bar{\alpha}^{o},t_f\right)B\bar{e}\left(\bar{\alpha}^{o},t_f\right) = \mathrm{col}(c_i) \qquad (i = 1, 2, \ldots m)$$

and

$$\delta\bar{v} = A\delta\bar{\alpha}^{o}$$

this transformation reduces

$$\left[\frac{\partial\bar{e}'}{\partial\bar{\alpha}}(\bar{\alpha}^o,t_f)B\,\frac{\partial\bar{e}}{\partial\bar{\alpha}}(\bar{\alpha}^o,t_f) - \lambda I\right]\delta\bar{\alpha}^o = -\,\frac{\partial\bar{e}'}{\partial\bar{\alpha}}(\bar{\alpha}^o,t_f)B\bar{e}(\bar{\alpha}^o,t_f)$$

through

$$A\left[\frac{\partial\bar{e}'}{\partial\bar{\alpha}}(\bar{\alpha}^o,t_f)B\,\frac{\partial\bar{e}}{\partial\bar{\alpha}}(\bar{\alpha}^o,t_f) - \lambda I\right]A'\delta\bar{v} = -A\,\frac{\partial\bar{e}'}{\partial\bar{\alpha}}(\bar{\alpha}^o,t_f)B\bar{e}(\bar{\alpha}^o,t_f)$$

to

$$\mathrm{diag}(\lambda_i - \lambda)\delta\bar{v} = \bar{c}$$

Because $\lambda < 0$

$$\delta\bar{v} = \mathrm{diag}\,\frac{1}{\lambda_i - \lambda}\,\bar{c}$$

and

$$\left\|\delta\bar{\alpha}^o\right\|^2 = \delta\bar{\alpha}^o \cdot \delta\bar{\alpha}^o = \delta\bar{v} \cdot \delta\bar{v} = \sum_{i=1}^{m} \frac{c_i^2}{\left(\lambda_i - \lambda\right)^2} = \nu^2$$

or

$$\sum_{i=1}^{m} \frac{c_i^2}{\left(\lambda_i + |\lambda|\right)^2} \tag{A2}$$

Assume that $\lambda_i \neq 0$ for all $i = 1, 2, \ldots m$. If

$$\nu^2 \geqq \sum_{i=1}^{m}\left(\frac{c_i}{\lambda_i}\right)^2$$

no real negative λ exists because the expression

$$\sum_{i=1}^{m} \frac{c_i^2}{\left(\lambda_i + |\lambda|\right)^2}$$

is strictly decreasing for increasing $|\lambda|$. For

$$\nu^2 < \sum_{i=1}^{m}\left(\frac{c_i}{\lambda_i}\right)^2$$

a unique λ which satisfies equation (A2) exists. If some c_i vanish, these terms in equation (A2) vanish independently of λ_i and the same arguments hold for the reduced equation. If, for $i = j$ $(j = 1, 2, \ldots m)$, $\lambda_j = 0$ but $c_j \neq 0$, then equation (A2) becomes

$$\frac{c_j^2}{|\lambda|^2} + \sum_{i \neq j, i=1}^{m} \frac{c_i^2}{\left(\lambda_i + |\lambda|\right)^2} = \nu^2$$

and solutions in λ exist for all ν. Finally, if all c_i vanish, a solution exists only for $\nu = 0$.

Lemma 2

The solutions $\delta\bar{\alpha}^o(\nu)$ and $\lambda(\nu) < 0$ of the system (eq. (A1)) maximize the absolute value of

$$\tilde{\Delta}E\left(\bar{\alpha}^o, t_f\right) = \frac{\partial \bar{e}'\left(\bar{\alpha}^o, t_f\right)}{\partial \bar{\alpha}} B\bar{e}\left(\bar{\alpha}^o, t_f\right) \cdot \delta\bar{\alpha}^o + \frac{1}{2}\, \delta\bar{\alpha}^o \cdot \frac{\partial \bar{e}'}{\partial \bar{\alpha}}\left(\bar{\alpha}^o, t_f\right) B\, \frac{\partial \bar{e}}{\partial \bar{\alpha}}\left(\bar{\alpha}^o, t_f\right) \delta\bar{\alpha}^o \qquad \text{(A3)}$$

subject to the conditions $\left\|\delta\bar{\alpha}^o\right\| \leqq \nu^2$ and $\tilde{\Delta}E\left(\bar{\alpha}^o, t_f\right) < 0$.

Proof: Note that the inequality condition on $\left\|\delta\bar{\alpha}^o\right\|$ can be replaced by an equality condition through the introduction of a real variable β because $\left\|\delta\bar{\alpha}^o\right\|^2 \leqq \nu^2$ implies and is implied by the existence of a β such that $\delta\bar{\alpha}^o \cdot \delta\bar{\alpha}^o - \nu^2 + \beta^2 = 0$. Then, $\left|\tilde{\Delta}E\left(\bar{\alpha}^o, t_f\right)\right|$ must be maximized with respect to the choice of $\delta\bar{\alpha}^o$ and β subject to

Condition (a):

$$\left\|\delta\bar{\alpha}^o\right\|^2 - \nu^2 + \beta^2 = 0$$

and

Condition (b):

$$\tilde{\Delta}E\left(\bar{\alpha}^o, t_f\right) < 0$$

A Lagrange multiplier λ (ref. 9) is introduced, and $\delta\bar{\alpha}^o$ and β are chosen such that the augmented relation

$$\left|\tilde{\Delta}E\left(\bar{\alpha}^o, t_f\right)\right|^* = \left|\tilde{\Delta}E\left(\bar{\alpha}^o, t_f\right)\right| + \frac{\lambda}{2}\left(\left\|\delta\bar{\alpha}^o\right\|^2 - \nu^2 + \beta^2\right)$$

is maximized. If condition (b) and equation (A3) are taken into account

$$\left|\tilde{\Delta}E\left(\bar{\alpha}^o, t_f\right)\right|^* = -\frac{\partial \bar{e}'}{\partial \bar{\alpha}}\left(\bar{\alpha}^o, t_f\right) B\bar{e}\left(\bar{\alpha}^o, t_f\right) \cdot \delta\bar{\alpha}^o - \frac{1}{2}\, \delta\bar{\alpha}^o \cdot \frac{\partial \bar{e}'}{\partial \bar{\alpha}}\left(\bar{\alpha}^o, t_f\right) B\, \frac{\partial \bar{e}}{\partial \bar{\alpha}}\left(\bar{\alpha}^o, t_f\right) \delta\bar{\alpha}^o$$

$$+ \frac{\lambda}{2}\left(\delta\bar{\alpha}^o \cdot \delta\bar{\alpha}^o - \nu^2 + \beta^2\right)$$

APPENDIX A

Necessary and sufficient conditions that $\left|\tilde{\Delta}E(\bar{\alpha}^o,t_f)\right|^*$ be maximized with respect to $\delta\bar{\alpha}^o$ and β are

Condition (c):

$$\frac{\partial\left|\tilde{\Delta}E(\bar{\alpha}^o,t_f)\right|^*}{\partial\bar{k}} = \bar{0}'$$

and

Condition (d):

$$\frac{\partial^2\left|\tilde{\Delta}E(\bar{\alpha}^o,t_f)\right|^*}{\partial\bar{k}^2} \text{ is negative definite}$$

where $\bar{k} = \begin{pmatrix}\delta\bar{\alpha}^o \\ \beta\end{pmatrix}$. Condition (c) yields the vector equation

$$\begin{pmatrix} -\frac{\partial\bar{e}'}{\partial\bar{\alpha}}(\bar{\alpha}^o,t_f)B\bar{e}(\bar{\alpha}^o,t_f) - \frac{\partial\bar{e}'}{\partial\bar{\alpha}}(\bar{\alpha}^o,t_f)B\frac{\partial\bar{e}}{\partial\bar{\alpha}}(\bar{\alpha}^o,t_f)\delta\bar{\alpha}^o + \lambda\delta\bar{\alpha}^o \\ \lambda\beta \end{pmatrix} = \begin{pmatrix}\bar{0} \\ 0\end{pmatrix}$$

and condition (d) yields the matrix condition

$$\begin{pmatrix} -\frac{\partial\bar{e}'}{\partial\bar{\alpha}}(\bar{\alpha}^o,t_f)B\frac{\partial\bar{e}}{\partial\bar{\alpha}}(\bar{\alpha}^o,t_f) + \lambda I, & \bar{0} \\ \bar{0}' & , \lambda \end{pmatrix} < 0$$

If A diagonalizes $\frac{\partial\bar{e}'}{\partial\bar{\alpha}}(\bar{\alpha}^o,t_f)B\frac{\partial\bar{e}}{\partial\bar{\alpha}}(\bar{\alpha}^o,t_f)$, then $G = \begin{pmatrix}A & \bar{0} \\ \bar{0}' & 1\end{pmatrix}$ diagonalizes $\frac{\partial^2\left|\tilde{\Delta}E(\bar{\alpha}^o,t_f)\right|^*}{\partial\bar{k}^2}$

or

$$G\frac{\partial^2\left|\tilde{\Delta}E(\bar{\alpha}^o,t_f)\right|^*}{\partial\bar{k}^2}G' = \begin{pmatrix}\operatorname{diag}(\lambda - \lambda_i), & \bar{0} \\ \bar{0}' & , \lambda\end{pmatrix}$$

Because G is nonsingular, examination of

$$G\frac{\partial^2\left|\tilde{\Delta}E(\bar{\alpha}^o,t_f)\right|^*}{\partial\bar{k}^2}G'$$

for negative definiteness is equivalent to the examination of

$$\frac{\partial^2\left|\tilde{\Delta}E(\bar{\alpha}^o,t_f)\right|^*}{\partial\bar{k}^2}$$

APPENDIX A

Because $\lambda_i \geqq 0$ $(i = 1, 2, \ldots m)$, $\dfrac{\partial^2 \left|\tilde{\Delta} E(\bar{\alpha}^o, t_f)\right|^*}{\partial \bar{k}^2}$ is negative definite for arbitrary λ_i if and only if $\lambda < 0$. From $\lambda\beta = 0$, $\beta = 0$, whereby

$$\delta\bar{\alpha}^o \cdot \delta\bar{\alpha}^o - \nu^2 + \beta^2 = 0$$

and

$$-\frac{\partial \bar{e}'}{\partial \bar{\alpha}} B\bar{e}(\bar{\alpha}^o, t_f) - \left[\frac{\partial \bar{e}'}{\partial \bar{\alpha}}(\bar{\alpha}^o, t_f) B \frac{\partial \bar{e}}{\partial \bar{\alpha}}(\bar{\alpha}^o, t_f) - \lambda I\right]\delta\bar{\alpha}^o = \bar{0}$$

yield equation (A1).

Lemma 3

The quantity $\tilde{\Delta} E(\bar{\alpha}^o, t_f)$, given by equation (A3), is negative definite if $\delta\bar{\alpha}^o$ for $\delta\bar{\alpha}^o$ satisfies equation (A1).

Proof: From equation (A1)

$$\delta\bar{\alpha}^o \cdot \frac{\partial \bar{e}'}{\partial \bar{\alpha}}(\bar{\alpha}^o, t_f) B\bar{e}(\bar{\alpha}^o, t_f) = \lambda\left(\delta\bar{\alpha}^o \cdot \delta\bar{\alpha}^o\right) - \delta\bar{\alpha}^o \cdot \frac{\partial \bar{e}'}{\partial \bar{\alpha}}(\bar{\alpha}^o, t_f) B \frac{\partial \bar{e}}{\partial \bar{\alpha}}(\bar{\alpha}^o, t_f)\, \delta\bar{\alpha}^o$$

which upon substitution into equation (A3) yields

$$\tilde{\Delta} E(\bar{\alpha}^o, t_f) = -|\lambda|\left(\delta\bar{\alpha}^o \cdot \delta\bar{\alpha}^o\right) - \frac{1}{2}\, \delta\bar{\alpha}^o \cdot \frac{\partial \bar{e}'}{\partial \bar{\alpha}}(\bar{\alpha}^o, t_f) B \frac{\partial \bar{e}}{\partial \bar{\alpha}}(\bar{\alpha}^o, t_f)\, \delta\bar{\alpha}^o$$

Thus, $\tilde{\Delta} E(\bar{\alpha}^o, t_f)$ is negative definite in $\delta\bar{\alpha}^o$ because $|\lambda| \neq 0$ and $\frac{\partial \bar{e}'}{\partial \bar{\alpha}}(\bar{\alpha}^o, t_f) B \frac{\partial \bar{e}}{\partial \bar{\alpha}}(\bar{\alpha}^o, t_f)$ is nonnegative definite. Therefore, $\tilde{\Delta} E(\bar{\alpha}^o, t_f) = 0$ if and only if $\delta\bar{\alpha}^o = 0$.

APPENDIX B

DYNAMIC EQUATIONS FOR LUNAR-RENDEZVOUS PROBLEM

Dynamic equations are developed for a space vehicle which seeks to rendezvous with a station in a circular orbit in the vicinity of the moon.

The vehicle is a one-stage rocket, treated as a point mass, with bounded thrust magnitude, and the controls are the magnitude and direction of the thrust vector.

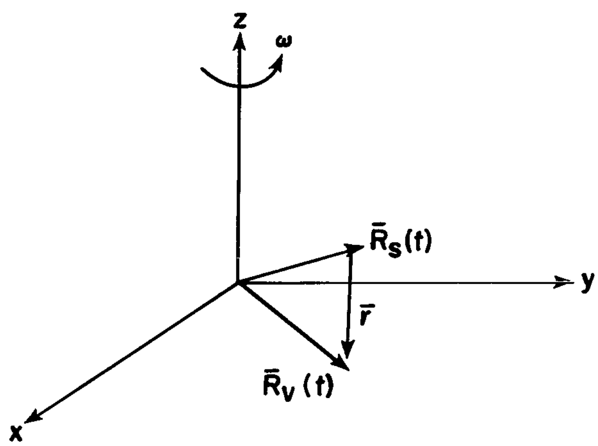

Figure B-1.- Rotating axis system.

Let x, y, and z be Cartesian coordinates of a rotating axis system located in the center of the moon with the z-axis through the axis of rotation of the moon. The geometry is represented in figure B-1.

The vector $\bar{R}_s(t)$ is from the origin to the space station. Let $\bar{R}_v(t)$ be the instantaneous vector from the origin to the vehicle and ω be the angular velocity of the moon about its axis of rotation. By assuming that

$$\frac{d}{dt} m(t) = -\frac{T}{c} \qquad \left(m(t_o) = m_o\right)$$

the dynamic equations of motion for the station and vehicle are

$$\left.\begin{aligned} \frac{d^2\bar{R}_v}{dt^2} &= \frac{\bar{T}}{m(t)} - \mu\frac{\bar{R}_v}{R_v{}^3} - 2\left(\bar{\omega}\times\frac{d\bar{R}_v}{dt}\right) - \bar{\omega}\times\left(\bar{\omega}\times\bar{R}_v\right) & \left(\bar{R}_v(t_o) = \bar{R}_v{}^o;\ \dot{\bar{R}}_v(t_o) = \dot{\bar{R}}_v{}^o\right) \\ \frac{d^2\bar{R}_s}{dt^2} &= -\mu\frac{\bar{R}_s}{R_s{}^3} - 2\left(\bar{\omega}\times\frac{d\bar{R}_s}{dt}\right) - \bar{\omega}\times\left(\bar{\omega}\times\bar{R}_s\right) & \left(\bar{R}_s(t_o) = \bar{R}_s{}^o;\ \dot{\bar{R}}_s(t_o) = \dot{\bar{R}}_s{}^o\right) \end{aligned}\right\} \tag{B1}$$

where

$\bar{\omega} = \hat{k}\omega$

$\bar{R}_s = \hat{i}R_{s_x} + \hat{j}R_{s_y} + \hat{k}R_{s_z}$

$\bar{R}_v = \hat{i}R_{v_x} + \hat{j}R_{v_y} + \hat{k}R_{v_z}$

$\hat{i},\hat{j},\hat{k}$ unit vectors along x-, y-, and z-axes, respectively

$m(t)$ total vehicle mass

μ universal gravitational constant multiplied by mass of moon

$R_V = \left(\overline{R}_V \cdot \overline{R}_V\right)^{1/2}$

$R_S = \left(\overline{R}_S \cdot \overline{R}_S\right)^{1/2}$

$\overline{T}$ thrust control vector of vehicle

T magnitude of $\overline{T}$

c effective exhaust velocity of vehicle rockets

The thrust vector is related to the xyz-axis system by

$$\overline{T} = \hat{i}\left(T \cos \theta_c \cos \varphi_c\right) + \hat{j}\left(T \cos \theta_c \sin \varphi_c\right) + \hat{k}\left(T \sin \theta_c\right)$$

as shown in figure B-2.

The vector $\overline{R}_S(t)$ can be found at any instant in the rotating coordinate system by integrating its differential equation with the appropriate initial conditions or by a mapping process. Because the satellite is assumed to be in a circular orbit, it moves in its orbital plane at a constant distance R_S from the center of the moon with a constant angular velocity $\Omega = \left(\mu/R_S{}^3\right)^{1/2}$. Consider an inertial XYZ-axis system fixed in the center of the moon such that, at the initial time t_o, it is alined with the rotating xyz-axis system. In this framework, the station can be pictured as in figure B-3.

The angles ι_o and θ_o define the normal and line of nodes, respectively, of the target orbital plane relative to the inertial system. The x' and y' axes define the orbital plane of the target. If, at t_o, the target is in the position $(x',y') = \left(R_S \cos \varphi_o, R_S \sin \varphi_o\right)$ and moves clockwise from $\overline{R}_S$, then

$$\overline{R}_S\left[x'(t),y'(t),z'(t)\right] = R_S \begin{Bmatrix} \cos\left[\varphi_o - \Omega(t - t_o)\right] \\ \sin\left[\varphi_o - \Omega(t - t_o)\right] \\ 0 \end{Bmatrix} \tag{B2}$$

Therefore

$$\overline{R}_S(X,Y,Z) = T_1\overline{R}_S(x',y',x') \tag{B3}$$

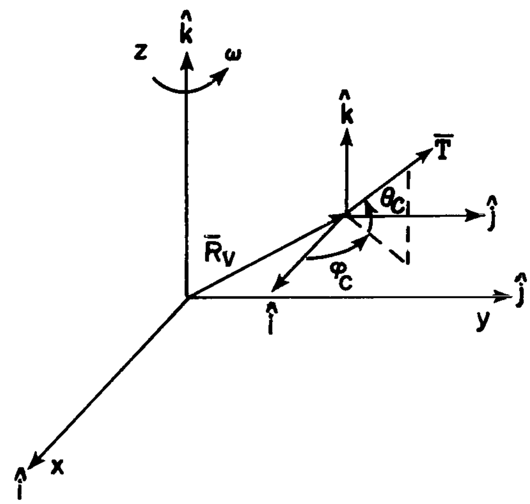

Figure B-2.- Reference axis system for control vector.

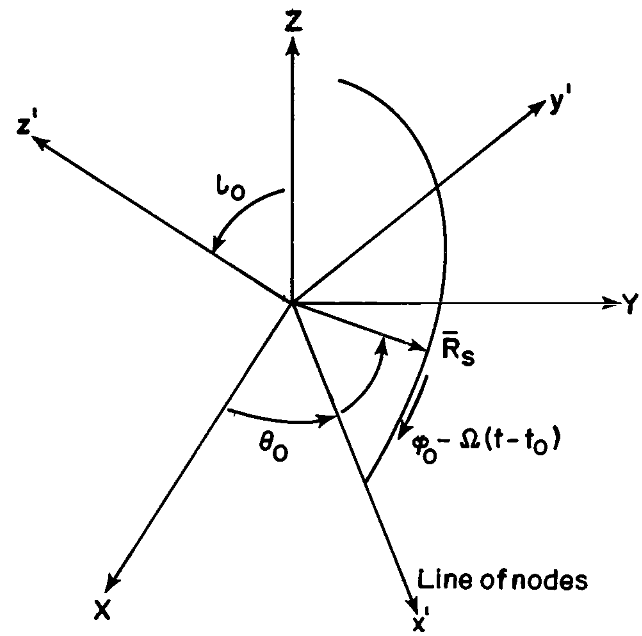

Figure B-3.- Station viewed in inertial axis system.

where

$$T_1 = \begin{bmatrix} \cos\theta_o & -\cos\iota_o \sin\theta_o & \sin\iota_o \sin\theta_o \\ \sin\theta_o & \cos\iota_o \cos\theta_o & -\sin\iota_o \cos\theta_o \\ 0 & \sin\iota_o & \cos\iota_o \end{bmatrix} \tag{B4}$$

Because the xyz-axis system rotates about the Z-axis with a constant angular velocity ω

$$\overline{R}_S(x,y,z) = \overline{R}_S(t) = T_2(t)\overline{R}_S(X,Y,Z)$$

where

$$T_2(t) = \begin{bmatrix} \cos\omega(t - t_o) & \sin\omega(t - t_o) & 0 \\ -\sin\omega(t - t_o) & \cos\omega(t - t_o) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{B5}$$

or

$$\overline{R}_S(t) = R_S\left(\begin{Bmatrix} \cos\left[\omega(t - t_o) - \theta_o\right] \\ -\sin\left[\omega(t - t_o) - \theta_o\right] \\ 0 \end{Bmatrix} \cos\left[\varphi_o - \Omega(t - t_o)\right]\right.$$

$$\left. + \begin{Bmatrix} \cos \iota_o \sin\left[\omega(t - t_o) - \theta_o\right] \\ \cos \iota_o \cos\left[\omega(t - t_o) - \theta_o\right] \\ \sin \iota_o \end{Bmatrix} \sin\left[\varphi_o - \Omega(t - t_o)\right]\right) \tag{B6}$$

Also

$$\frac{d\overline{R}_S(t)}{dt} = \dot{T}_2(t)R_S(X,Y,Z) + T_2(t)\dot{\overline{R}}_S(X,Y,Z)$$

with

$$\dot{T}_2(t) = \omega\begin{bmatrix} -\sin \omega(t - t_o) & \cos \omega(t - t_o) & 0 \\ -\cos \omega(t - t_o) & -\sin \omega(t - t_o) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\dot{\overline{R}}_S(X,Y,Z) = R_S\Omega T_1\begin{Bmatrix} \sin\left[\varphi_o - \Omega(t - t_o)\right] \\ -\cos\left[\varphi_o - \Omega(t - t_o)\right] \\ 0 \end{Bmatrix}$$

or

$$\frac{d\overline{R}_S(t)}{dt} = R_S\left(\begin{Bmatrix} -\sin\left[\omega(t - t_o) - \theta_o\right]\left(\omega + \Omega \cos \iota_o\right) \\ -\cos\left[\omega(t - t_o) - \theta_o\right]\left(\omega + \Omega \cos \iota_o\right) \\ -\Omega \sin \iota_o \end{Bmatrix} \cos\left[\varphi_o - \Omega(t - t_o)\right]\right.$$

(Equation continued on next page)

$$+\begin{Bmatrix} \cos\left[\omega(t - t_o) - \theta_o\right](\omega \cos \iota_o + \Omega) \\ -\sin\left[\omega(t - t_o) - \theta_o\right](\omega \cos \iota_o + \Omega) \\ 0 \end{Bmatrix} \sin\left[\varphi_o - \Omega(t - t_o)\right]\Bigg) \tag{B7}$$

Thus, the position and rate of $\overline{R}_S(t)$ can be obtained by specifying R_S, ι_o, θ_o, and φ_o at t_o and by using equations (B6) and (B7). The initial value $\overline{R}_V(t_o)$ can be specified by

$$\overline{R}_V(t_o) = R_V(t_o)\left(\hat{i} \cos \theta_V{}^o \cos \varphi_V{}^o + \hat{j} \cos \theta_V{}^o \sin \varphi_V{}^o + \hat{k} \sin \theta_V{}^o\right)$$

where $\theta_V{}^o$ and $\varphi_V{}^o$ are shown in figure B-4. Also

$$\dot{\overline{R}}_V(t_o) = \hat{i}\dot{R}_{V_x}(t_o) + \hat{j}\dot{R}_{V_y}(t_o) + \hat{k}\dot{R}_{V_z}(t_o) \tag{B8}$$

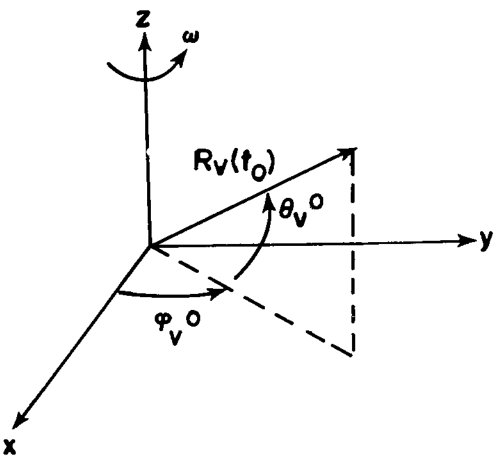

Figure B-4.- Initial orientation of vehicle with respect to rotating axis system.

Rewriting equation (B1) in vector-matrix notation yields

$$\left.\begin{aligned} \ddot{\overline{R}}_V &= \frac{u_4\hat{u}}{m(t)} - \mu\,\frac{\overline{R}_V}{R_V{}^3} - 2S\dot{\overline{R}}_V - S^2\overline{R}_V \\ &\qquad \left(\overline{R}_V(t_o) = \overline{R}_V{}^o;\ \dot{\overline{R}}_V(t_o) = \dot{\overline{R}}_V{}^o\right) \\ \ddot{\overline{R}}_S &= -\mu\,\frac{\overline{R}_S}{R_S{}^3} - 2S\dot{\overline{R}}_S - S^2\overline{R}_S \\ &\qquad \left(\overline{R}_S(t_o) = \overline{R}_S{}^o;\ \dot{\overline{R}}_S(t_o) = \dot{\overline{R}}_S{}^o\right) \end{aligned}\right\} \tag{B9}$$

where

$$\overline{R}_S = \begin{bmatrix} R_{S_x} \\ R_{S_y} \\ R_{S_z} \end{bmatrix}$$

$$\overline{R}_V = \begin{bmatrix} R_{V_x} \\ R_{V_y} \\ R_{V_z} \end{bmatrix}$$

$$\hat{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \cos\theta_c \cos\varphi_c \\ \cos\theta_c \sin\varphi_c \\ \sin\theta_c \end{bmatrix}$$

$$u_4 = T$$

and

$$S = \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In terms of the relative distance $\bar{r} = \overline{R}_V - \overline{R}_S$

$$\ddot{\bar{r}} = \frac{u_4\hat{u}}{m(t)} - \Omega^2\bar{r} - \Omega^2\left(\bar{r} + \overline{R}_S\right)\left(\frac{R_S{}^3}{\left\|\bar{r} + \overline{R}_S\right\|^3} - 1\right) - 2S\dot{\bar{r}} - S^2\bar{r} \tag{B10}$$

where

$$\left\|\bar{r} + \overline{R}_S\right\|^2 = \left(\bar{r} + \overline{R}_S\right) \cdot \left(\bar{r} + \overline{R}_S\right)$$

Because the maximum principle is used for optimization purposes in appendix C, the state-vector notation is now employed. Let

$$x_1 = r_x$$
$$x_2 = \dot{r}_x = \dot{x}_1$$
$$x_3 = r_y$$
$$x_4 = \dot{r}_y = \dot{x}_3$$
$$x_5 = r_y$$
$$x_6 = \dot{r}_z = \dot{x}_5$$
$$x_7 = m(t)$$

and

$$x_8 = t$$

With $\overline{R}_S$ and $\dot{\overline{R}}_S$ regarded as explicit functions of time by equations (B6) and (B7), write

$$\left.\begin{aligned} \dot{\bar{v}} &= \frac{u_4 M\hat{u}}{x_7} + \bar{Y}(\bar{v},x_8) && \left(\bar{v}(t_o) = \bar{v}_o;\ \ \bar{v}(t_f) = \bar{0}\right) \\ \dot{x}_7 &= -\frac{u_4}{c} && \left(x_7(t_o) = m(t_o)\right) \\ \dot{x}_8 &= 1 && \left(x_8(t_o) = t_o\right) \end{aligned}\right\} \tag{B11}$$

where

$$\bar{Y}(\bar{v},x_8) = -\frac{\Omega^2 R_s{}^3}{\left\|A\bar{v} + R_s\right\|^3}\left[N\bar{v} + M\bar{R}_s(x_8)\right] + \Omega^2 M\bar{R}_s(x_8) + \left(N' + 2\omega K + \omega^2 L\right)\bar{v}$$

$\bar{v} = \mathrm{col}(x_1, \ . \ . \ . \ x_6)$

t_o launch time

t_f final rendezvous time

$$M = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$N = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The act of rendezvous requires that the vehicle and station have the same position and velocity at t_f; hence, the condition $\bar{v}(t_f) = \bar{0}$. In addition, $u_4 \leqq \gamma$, where γ is the largest value obtainable for the thrust magnitude.

APPENDIX C

NECESSARY CONDITIONS FOR FUEL-OPTIMAL RENDEZVOUS

Given the system of equations (B11), establish necessary conditions that the control functions $\hat{u}$ and u_4 drive $\bar{v}(t)$ from $\bar{v}(t_o)$ to $\bar{v}(t_f) = \bar{0}$ in such a way as to minimize $x_o(t_f) \triangleq \int_{t_o}^{t_f} \frac{u_4}{c}\, dt = m(t_o) - m(t_f)$. These conditions readily follow from the Pontryagin maximum principle (ref. 4).

Before the maximum principle can be stated for this particular problem, the variables x_o, $\underset{\sim}{H}$, and ψ_k $(k = 0, 1, \ldots 8)$ must be defined as follows:

$$\left.\begin{aligned} \dot{x}_o &= \frac{u_4}{c} \qquad\qquad \left(x_o(t_o) = 0\right) \\ \underset{\sim}{H} &= \sum_{k=0}^{8} \psi_k \dot{x}_k \\ \dot{\psi}_k &= -\frac{\partial \underset{\sim}{H}}{\partial x_k} \end{aligned}\right\} \tag{C1}$$

Through equations (B11), the equations for ψ_k $(k = 0, 1, \ldots 8)$ are

$$\begin{aligned} \dot{\psi}_o &= 0 \\ \dot{\bar{h}} &= -\frac{\partial \bar{Y}'\left(\bar{v}, x_8\right)}{\partial \bar{v}} \bar{h} \\ \dot{\psi}_7 &= \frac{u_4 \bar{h} \cdot M\hat{u}}{x_7^2} \\ \dot{\psi}_8 &= -\frac{\partial \bar{Y}\left(\bar{v}, x_8\right) \cdot \bar{h}}{\partial x_8} \end{aligned}$$

where

$$\bar{h} = \mathrm{col}\left(\psi_1, \psi_2, \ldots \psi_6\right)$$

The Pontryagin maximum principle can then be stated as follows: Let u_i $(i = 1, 2, \ldots 4)$, where $\sum_{i=1}^{3} u_i^2 \equiv 1$ and $0 \leqq u_4 \leqq \gamma$, be controls which transfer $x_j(t_o)$

to $x_j(t_f)$ $(j = 0, 1, \ldots 8)$. In order that u_i minimize $x_o(t_f)$, it is necessary that there exist a nonzero continuous vector with elements ψ_j $(j = 0, 1, \ldots 8)$ as determined by equation (C1) such that:

(1) For every t $(t_o \leqq t \leqq t_f)$, the function $\underset{\sim}{H}(x_j, \psi_j, u_i)$, for fixed x_j and ψ_j, attains its maximum $\underset{\sim}{M}(x_j, \psi_j)$ at the point $u_i = u_i(t)$; that is, $\underset{\sim}{H}\left[\psi_j, x_j, u_i(t)\right] = \underset{\sim}{M}(\psi_j, x_j)$. If equations (B11), (C1), and $\underset{\sim}{H}\left[\psi_j, x_j, u_i(t)\right] = \underset{\sim}{M}(\psi_j, x_j)$ are satisfied, then $\psi_o \leqq 0$ and $\underset{\sim}{M}\left[x_j(t), \psi_j(t)\right] = 0$.

(2) Because $x_7(t_f)$ and $x_8(t_f)$ are unconstrained, the part of the maximum principle known as the transversality condition requires that $\psi_7(t_f) = \psi_8(t_f) = 0$. Substitution of $\dot{x}_j$ $(j = 0, 1, \ldots 8)$ into $\underset{\sim}{H}$ gives the equation

$$\underset{\sim}{H} = u_4\left(\frac{\hat{u} \cdot M'\bar{h}}{x_7} - \frac{\psi_7}{c} + \frac{\psi_o}{c}\right) + \bar{h} \cdot \overline{Y}(\bar{v}, x_8) + \psi_8$$

If $M'\bar{h} \neq 0$, then the $\hat{u}$ which maximizes $\underset{\sim}{H}$ and satisfies $\hat{u} \cdot \hat{u} \equiv 1$ is $\hat{u} = \dfrac{M'\bar{h}}{\left\|M'\bar{h}\right\|}$. The assumption $M'\bar{h} \equiv 0$ over a finite interval in $\left[t_o, t_f\right]$ leads to the contradiction $\psi_j(t_f) = 0$ $(j = 0, 1, \ldots 8)$ and, therefore, cannot occur on an optimal trajectory. If $M'\bar{h} = \bar{0}$ at isolated points of $\left[t_o, t_f\right]$, then the continuity of $\psi_j(t)$ implies that, if t' is such a point, $\hat{u}(t') = \dfrac{M'\bar{h}(t'^-)}{\left\|M'\bar{h}(t'^-)\right\|}$. Then, by using the optimal direction $\hat{u} = \dfrac{M'\bar{h}}{\left\|M'\bar{h}\right\|}$, $\underset{\sim}{H}$ becomes $u_4\left(\dfrac{\left\|M'\bar{h}\right\|}{x_7} - \dfrac{\psi_7}{c} + \dfrac{\psi_o}{c}\right) + \bar{h} \cdot \overline{Y}(\bar{v}, x_8) + \psi_8$. The u_4 which maximizes $\underset{\sim}{H}$, if $\rho(t) = \dfrac{\left\|M'\bar{h}\right\|}{x_7} - \dfrac{\psi_7}{c} + \dfrac{\psi_o}{c}$ vanishes only at isolated points within $\left[t_o, t_f\right]$, is

$$u_4 = \begin{cases} \gamma & (\rho > 0) \\ 0 & (\rho < 0) \end{cases}$$

or

$$u_4 = \frac{\gamma}{2}\left[1 + \operatorname{sgn} \rho\right] \tag{C2}$$

Situations in which $\rho \equiv 0$ over a finite interval in $\left[t_o, t_f\right]$ are referred to as being singular. Such cases may always occur in a general problem when the control enters linearly in $\underset{\sim}{H}$. In this case, the coefficient of the control must be examined to determine whether there is an admissible control rendering it identically zero. Such a control is termed a singular control. The existence of a singular control brings into account the difficult question of uniqueness of optimal controls. Necessary conditions for singular controls to be optimal are given in reference 11.

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In general, the existence and optimality of a singular control u_4 rendering $\rho \equiv 0$ within $[t_o, t_f]$ should be examined. However, because the purpose of the fuel-optimal-rendezvous problem is to exemplify the use of the algorithm in the solution of a class of two-point boundary-value problems, singular solutions are not considered. All solutions obtained are for the nonsingular control law

$$\bar{u} = u_4 \hat{u} = \frac{\gamma}{2}\left[1 + \operatorname{sgn} \rho\right] \frac{M'\bar{h}}{\|M'\bar{h}\|} \tag{C3}$$

Substitution of $\bar{u}$ into $\underset{\sim}{H}$ yields

$$\underset{\sim}{M}(x_j, \psi_j) = \frac{\gamma}{2}\left[1 + \operatorname{sgn} \rho\right]\rho + \bar{h} \cdot \bar{Y}(\bar{v}, x_8) + \psi_8$$

At t_f, where $\bar{v}(t_f) = \bar{0}$ and $\psi_8(t_f) = 0$, $\underset{\sim}{M}(t_f) = 0$ yields

$$\frac{\gamma}{2}\left[1 + \operatorname{sgn} \rho\right]\rho(t_f) = 0$$

Thus, at t_f, $\rho(t_f) \leqq 0$. Because, theoretically, a coast into a rendezvous cannot be performed, $\rho(t_f) = 0$ $\left(\dot{\rho}(t_f) \leqq 0\right)$. This property classifies t_f; namely

$$t_f \epsilon \left[t; \rho(t) = 0, \dot{\rho}(t) \leqq 0\right]$$

The equations to be satisfied along an optimal trajectory take the form

$$\left.\begin{aligned}
&\dot{x}_o = \frac{\gamma}{2c}\left[1 + \operatorname{sgn} \rho\right] && \left(x_o(t_o) = 0\right) \\
&\dot{\bar{v}} = \frac{\gamma}{2}\left[1 + \operatorname{sgn} \rho\right] \frac{MM'\bar{h}}{\|M'\bar{h}\| x_7} - \frac{\Omega^2 R_s^{\ 3}}{\|A\bar{v} + \bar{R}_s\|^3}\left[N\bar{v} + M\bar{R}_s(x_8)\right] + \Omega M \bar{R}_s(x_8) && \\
&\qquad + \left(N' + 2\omega K + \omega^2 L\right)\bar{v} && \left(\bar{v}(t_o) = \bar{v}_o;\ \ \bar{v}(t_f) = \bar{0}\right) \\
&\dot{x}_7 = -\frac{\gamma}{2c}\left[1 + \operatorname{sgn} \rho\right] && \left(x_7(t_o) = m_o\right) \\
&\dot{x}_8 = 1 && \left(x_8(t_o) = t_o\right) \\
&\dot{\psi}_o = 0 && \left(\psi_o(t_o) \leqq 0\right) \\
&\dot{\bar{h}} = \frac{\Omega^2 R_s^{\ 3} N'\bar{h}}{\|A\bar{v} + \bar{R}_s\|^3} - \frac{3\Omega^2 R_s^{\ 3}\left\{\left[N\bar{v} + M\bar{R}_s(x_8)\right] \cdot \bar{h}\right\} A'}{\|A\bar{v} + \bar{R}_s\|^5}\left[A\bar{v} + \bar{R}_s(x_8)\right] && \\
&\qquad - \left(N + 2\omega K' + \omega^2 L'\right)\bar{h} && \left(\bar{h}(t_o)\ \ \text{undetermined}\right)
\end{aligned}\right\} \tag{C4}$$

(Equations continued on next page)

$$\left.\begin{aligned}
\dot{\psi}_7 &= \frac{\gamma}{2}\left[1 + \operatorname{sgn}\rho\right]\frac{\|M'\bar{h}\|}{x_7^2} && \left(\psi_7(t_f) = 0\right) \\
\dot{\psi}_8 &= \frac{\Omega^2 R_s^3 \bar{h} \cdot M\bar{R}_s(x_8)}{\|A\bar{v} + \bar{R}_s\|^3} - \Omega^2 \bar{h} \cdot M\dot{\bar{R}}_s(x_8) \\
&\quad - \frac{3\Omega^2 R_s^3}{\|A\bar{v} + \bar{R}_s\|^5} \dot{\bar{R}}_s \cdot \left(A\bar{v} + \bar{R}_s\right)\left\{\bar{h} \cdot \left[N\bar{v} + M\bar{R}_s(x_8)\right]\right\} && \left(\psi_8(t_f) = 0\right)
\end{aligned}\right\} \tag{C4}$$

with

$$\rho = \frac{\|M'\bar{h}\|}{x_7} - \frac{\psi_7}{c} + \frac{\psi_o}{c} \qquad \left(\rho(t_f) = 0;\ \dot{\rho}(t_f) \leqq 0\right)$$

Because t_o is interpreted to be the launch time

$$\psi_8(t_o) = -\left[\gamma\rho(t_o) + \bar{h}(t_o) \cdot \bar{\gamma}(\bar{v}_o, t_o)\right]$$

implies that $\underset{\sim}{M}(t_o) = 0$ whereby $\underset{\sim}{M}(t) \equiv 0$ on $\left[t_o, t_f\right]$ which guarantees that $\psi_8(t_f) = 0$. Thus, $\psi_8(t_o)$ can be determined to eliminate the condition $\psi_8(t_f) = 0$. If x_8 is replaced by t, then the variables x_o, x_8, and ψ_8 can be eliminated and computed, if desired, after $\bar{v}(t)$, $x_7(t)$, ψ_o, $\psi_7(t)$, and $\bar{h}(t)$ have been found.

Thus, a two-point boundary-value problem occurs which requires the determination of $\psi_j(t_o)$ $(j = 0, 1, \ldots 7)$ such that at t_f, $x_j(t_f) = 0$ $(j = 1, 2, \ldots 6)$ and $\psi_7(t_f) = 0$. A reduction in the number of parameters $\psi_j(t_o)$ and terminal conditions can be made by observing that the combination $\psi_7(t) - \psi_o$ satisfies the same differential equation as $\psi_7(t)$ and, therefore, can be computed collectively. If a value of $\psi_7(t_o) - \psi_o$ can be set for which $\psi_j(t_o)$ $(j = 1, 2, \ldots 6)$ can be found such that $\bar{v}(t_f) = 0$, but $\psi_7(t_f) \neq 0$ necessarily, then new values of $\psi_7(t_o)$ and ψ_o can be computed for which $\psi_7(t_f) = 0$ and the combination remains the same. These new values are found by replacing $\psi_7(t_o)$ by $\psi_7(t_o) - \psi_7(t_f)$ and ψ_o by $\psi_o - \psi_7(t_f)$. From $\rho(t_f) = 0$, $\psi_o = -\dfrac{c\|M'\bar{h}(t_f)\|}{x_7(t_f)}$ and verifies that $\psi_o \leqq 0$.

Finally, the problem is to find $\bar{h}(t_o)$ such that $\bar{v}(t_f) = \bar{0}$ with $\psi_7(t_o) - \psi_o$ normalized and $t_f \epsilon \left[t; \rho(t) = 0, \dot{\rho}(t) \leqq 0\right]$. Such a solution yields a trajectory which satisfies the necessary conditions for free-final-time fuel optimality.

In conclusion, if t_f is fixed, $\psi_8(t_f)$ does not necessarily equal zero (ref. 4) in which case $\psi_8(t_f)$ can be adjusted to satisfy $\underset{\sim}{M}(t_f) = 0$ and, thus, eliminate the necessity of $\rho(t_f) = 0$. If a solution can be obtained with $\rho(t_f)$ arbitrary but $\psi_o \leqq 0$, then the trajectory satisfies the necessary conditions of the Pontryagin maximum principle for fixed-final-time fuel optimality.

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